

# Analytical Solution for a Vortex on a Spherical Surface

G.D. Thiar<sup>a</sup>

Received 13 February 2009 and accepted 22 December 2009

*An analytical stream function solution for vortical fluid motion on a spherical surface over prescribed latitudinal and longitudinal extents is developed. The purpose of this solution is to provide initial conditions for computational fluid dynamic simulations of global ocean circulation, in order to avoid or reduce the computation time expended in spinning up a simulation from rest. A case study is provided to illustrate that realistic flow patterns, with westward intensification of current strength, can be generated. Some guidelines for extending the solution to situations for which it is not explicitly designed are given.*

## Nomenclature

### Roman letters

$A_m$	homogeneous solution constant
$C_p$	pressure coefficient
$i, j, k, \ell, m, n$	indices
$I_{m,n}, J_{m,n}$	special integrals
$p$	static pressure
$Q_0^n$	zero-order associated Legendre functions of the second kind
$r$	radial coordinate
$R$	radius
$\vec{V}$	translational velocity vector
$V_\varphi, V_\theta$	longitudinal and latitudinal velocity components
$z$	axial coordinate

### Greek

$\vec{\nabla}$	gradient operator
$\rho$	density
$\varphi, \theta$	longitudinal and latitudinal coordinates
$\eta, \xi$	transformed longitudinal and latitudinal coordinates
$\psi$	stream function (total)
$\tilde{\psi}$	stream function (homogeneous solution, dimensionless)
$\hat{\psi}$	stream function (particular solution, dimensionless)
$\Psi_{ij}$	particular solution coefficient
$\vec{\omega}$	vorticity vector
$\omega_r, \tilde{\omega}_r$	radial vorticity component, dimensionless radial vorticity component
$\Omega_{ij}$	radial vorticity coefficient
$\vec{\Omega}, \Omega$	angular velocity vector and magnitude

<sup>a</sup> SAIMEchE member  
Department of Mechanical and Mechatronic Engineering  
University of Stellenbosch  
Private Bag X1, Matieland, 7602.  
E-mail: thiar@sun.ac.za

## Subscripts

0	reference value
W	western boundary
E	eastern boundary

## 1. Introduction

The utilization of ocean currents as a renewable energy source has received much less attention than solar, wind and even wave energy. Nevertheless, ocean current devices may become important in the not-too-distant future, especially since the available energy density seems to be at least on par with that of wind energy for certain locations along the world's coastlines: according to a recent US white paper<sup>1</sup> "The total worldwide power in ocean currents has been estimated to be about 5,000 GW, with power densities of up to 15 kW/m<sup>2</sup> ...".

Efficient placement of ocean current energy extracting devices along any particular coastline would depend to a large extent on understanding the dynamics of these currents and to what extent they will be influenced by the presence of such devices. One way to study ocean current dynamics is by means of computation fluid dynamic (CFD) analysis, in which the flow of a variable density, variable salinity fluid on a rotating (essentially) spherical surface is simulated numerically; see for example the monograph by Miller<sup>2</sup>. There is a number of limiting factors which inhibit progress along this avenue, not the least of which is the number of cells that is required for an efficacious simulation: a modest size ocean covering one tenth of the earth's surface and which is about 4 km deep would require approximately  $2 \times 10^9$  cells if the mesh spacing is about 100 m in the depth direction and about ten times larger (i.e. 1 km) in the latitudinal and longitudinal directions. In addition CFD simulations usually have to be "spun up" from a stationary earth situation (see for example chapter 11 in the monograph by Pond and Pickard<sup>3</sup>), on completion of which process conservation of mass, momentum and energy have to be achieved before there can be proceeded with further investigations. Such a spin-up process can be quite (CPU) time consuming, and realistic initial conditions that obviate the need for it would be very convenient. It is the purpose of this paper to present an analytical solution that can be used to provide such initial conditions, if only to reduce the CPU time of the spin-up phase. It must also be mentioned here that commercial CFD software, such as ANSYS FLUENT, cannot be "started up" with zero flow everywhere for simulations such as these, where there are no inflow and outflow boundaries; initial conditions of some sort are essential.

There appears to be very few known analytical solutions for fluids rotating in non-circular containers under the influence of Coriolis forces. In his groundbreaking paper Stommel<sup>4</sup> presented a linearized analytical solution for the wind-driven circulation in a homogeneous rectangular ocean without curvature. In this solution the effects of bottom friction, horizontal pressure gradients caused by variable

surface height and Coriolis force were included by means of appropriate models. Other authors usually developed their mathematical models along the same lines as that of Stommel, e.g. Fofonoff<sup>5</sup>, who neglected viscous effects but included nonlinear inertia terms. In the present paper an analytical stream function solution for vortical fluid motion on a *spherical surface* extending south or north from the equator over given latitudinal and longitudinal extents is developed and demonstrated.

**2. Problem Statement**

We want to obtain the velocity field  $\vec{V}$  and the corresponding pressure field  $p$  for an ideal (incompressible, inviscid) fluid which executes a steady vortex-like motion on a rotating spherical surface with radius  $R$ , in a sector bounded by two latitudes, one being the equator of the spherical surface, and two longitudes. It is natural, therefore, to work in spherical coordinates  $(r, \theta, \varphi)$ , where  $\theta$  denotes the latitude and  $\varphi$  the longitude, as illustrated in figure 1. For this specific case, then,  $\vec{V} = V_\theta(\theta, \varphi)\vec{e}_\theta + V_\varphi(\theta, \varphi)\vec{e}_\varphi$  is the velocity *relative* to the rotating surface, and  $p = p(\theta, \varphi)$  is the pressure relative to the hydrostatic pressure.

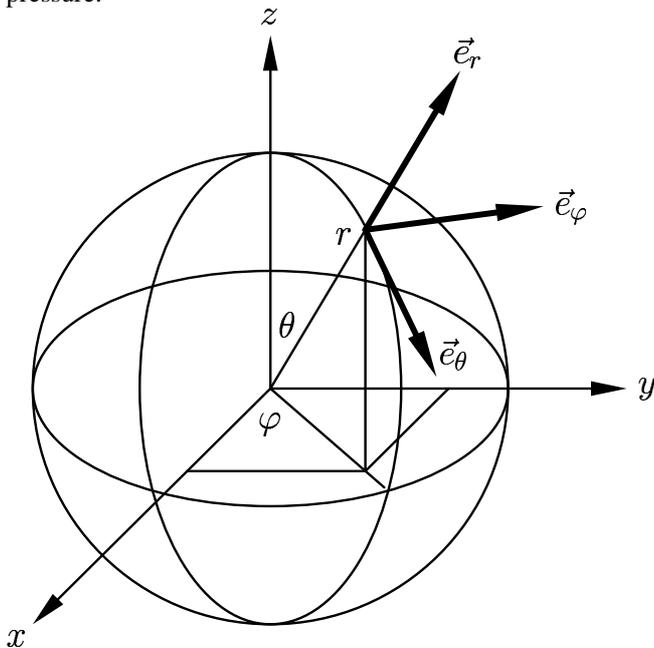


Figure 1: Spherical co-ordinates

**2.1 Governing equations and boundary conditions**

The governing equations for the steady motion of an ideal fluid are the continuity equation (which expresses the conservation of mass) and Euler’s equation of motion (which expresses the conservation of momentum) - see for example Karamcheti<sup>6</sup>. For the case under consideration, these two equations can be expressed as follows:

$$\text{Continuity : } \vec{\nabla} \cdot \vec{V} = \frac{\partial}{\partial \theta} (\sin \theta V_\theta) + \frac{\partial V_\varphi}{\partial \varphi} = 0 \tag{1}$$

$$\text{Euler : } \frac{1}{\rho} \vec{\nabla} p = -\frac{1}{2} \vec{\nabla} (\vec{V} \cdot \vec{V}) + \vec{V} \times (\vec{\omega} + 2\vec{\Omega}) \tag{2}$$

In equation 2  $\vec{\omega} = \nabla \times \vec{V}$  denotes the vorticity of the flow, and  $\vec{\Omega} = \Omega \vec{e}_z$  denotes the angular velocity of the rotating surface.

Equations 1-2 have to be solved subject to the following (southern hemisphere biased) conditions for no flow across the boundaries:

$$\begin{aligned} \varphi = \varphi_0, \varphi = \varphi_0 + \Delta\varphi : V_\theta = 0 \\ \text{and } \theta = \frac{\pi}{2}, \theta = \frac{\pi}{2} + \Delta\theta : V_\varphi = 0 \end{aligned} \tag{3}$$

In addition, it is necessary to specify the “strength” of the vortex. For the purpose of the analytical solution, i.e. to serve as initial condition for various CFD simulations, it is convenient to specify the values of maximum (in magnitude) velocities at the “western” and “eastern” boundaries, and to enforce the condition that they achieve maximum values at these boundaries:

$$\begin{aligned} \varphi = \varphi_0, \theta = \frac{\pi}{2} + \theta_W : V_\theta = V_W, \frac{\partial V_\theta}{\partial \theta} = \frac{\partial V_\theta}{\partial \varphi} = 0 \\ \varphi = \varphi_0 + \Delta\varphi, \theta = \frac{\pi}{2} + \theta_E : V_\theta = V_E, \frac{\partial V_\theta}{\partial \theta} = \frac{\partial V_\theta}{\partial \varphi} = 0 \end{aligned} \tag{4}$$

**2.2 Stream function formulation**

It is convenient to define a stream function  $\psi$  such that

$$V_\theta = -\frac{1}{R \sin \theta} \frac{\partial \psi}{\partial \varphi} \text{ and } V_\varphi = \frac{1}{R} \frac{\partial \psi}{\partial \theta} \tag{5}$$

This definition is such that the continuity equation is satisfied, as is easily verified by inserting the two velocity component definitions of equation 5 into equation 1. The radial component of the vorticity associated with this definition of stream function is

$$\begin{aligned} \omega_r &= \frac{1}{R^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} (R \sin \theta V_\varphi) - \frac{\partial}{\partial \varphi} (R V_\theta) \right] \\ &= \frac{1}{R^2 \sin^2 \theta} \left[ \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta - \frac{\partial \psi}{\partial \theta} + \frac{\partial^2 \psi}{\partial \varphi^2} \right) \right] \end{aligned}$$

or, expressed as a partial differential equation for  $\psi$ ,

$$\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{\partial^2 \psi}{\partial \varphi^2} = R^2 \sin^2 \theta \omega_r(\theta, \varphi) \tag{6}$$

The stream function boundary conditions that are consistent with the velocity boundary conditions stipulated by equation 3 are simply that the stream function has to have the same constant value on all boundaries; for the sake of convenience we take the constant to be zero, so that the stream function boundary conditions are

$$\varphi = \varphi_0, \varphi = \varphi_0 + \Delta\varphi, \theta = \frac{\pi}{2}, \theta = \frac{\pi}{2} + \Delta\theta : \psi = 0 \tag{7}$$

and the vortex strength conditions are

$$\begin{aligned} \varphi = \varphi_0, \quad \theta = \frac{\pi}{2} + \theta_W : \quad \frac{\partial \psi}{\partial \varphi} = -RV_W \cos(\theta_W), \quad \frac{\partial^2 \psi}{\partial \varphi^2} = 0, \\ \frac{\partial^2 \psi}{\partial \varphi \partial \theta} = RV_W \sin(\theta_W) \\ \varphi = \varphi_0 + \Delta \varphi, \quad \theta = \frac{\pi}{2} + \theta_E : \quad \frac{\partial \psi}{\partial \varphi} = -RV_E \cos(\theta_E), \quad \frac{\partial^2 \psi}{\partial \varphi^2} = 0, \\ \frac{\partial^2 \psi}{\partial \varphi \partial \theta} = RV_E \sin(\theta_E) \end{aligned} \quad (8)$$

Equation 6 is in the form of Poisson's equation, the solution of which consists of the homogeneous solution, i.e. the solution with the right hand side set to zero, and the particular solution. In order to simplify the solution procedure we introduce the following transformation:

$$\xi = \cos \theta, \eta = \frac{\pi(\varphi - \varphi_0)}{\Delta \varphi}, \psi = \Omega R^2 (\tilde{\psi} + \hat{\psi}) \quad (9)$$

and  $\omega_r = \Omega \tilde{\omega}_r$ ,

where  $\tilde{\psi}$  denotes the dimensionless homogeneous solution,  $\hat{\psi}$  the dimensionless particular solution, and  $\tilde{\omega}_r$  the dimensionless radial vorticity. Then equation 6 transforms as follows:

$$\begin{aligned} (1 - \xi^2) \frac{\partial}{\partial \xi} \left[ (1 - \xi^2) \frac{\partial}{\partial \xi} (\tilde{\psi} + \hat{\psi}) \right] + \left( \frac{\pi}{\Delta \varphi} \right)^2 \frac{\partial^2}{\partial \eta^2} (\tilde{\psi} + \hat{\psi}) \\ = (1 - \xi^2) \tilde{\omega}_r(\xi, \eta) \end{aligned} \quad (10)$$

The transformed boundary conditions are

$$\eta = 0, \quad \eta = \pi, \quad \xi = 0, \quad \xi = -\sin(\Delta \theta): \quad \tilde{\psi} + \hat{\psi} = 0 \quad (11)$$

and the transformed vortex strength conditions are

$$\begin{aligned} \eta = 0, \quad \xi = \xi_W = -\sin(\theta_W): \\ \frac{\partial^2}{\partial \eta} (\tilde{\psi} + \hat{\psi}) = -\frac{V_W}{k\Omega R} \cos(\theta_W), \quad \frac{\partial^2}{\partial \eta^2} (\tilde{\psi} + \hat{\psi}) = 0 \\ \frac{\partial^2}{\partial \xi \partial \eta} (\tilde{\psi} + \hat{\psi}) = -\frac{V_W}{k\Omega R} \tan(\theta_W) \\ \eta = \pi, \quad \xi = \xi_E = -\sin(\theta_E): \\ \frac{\partial^2}{\partial \eta} (\tilde{\psi} + \hat{\psi}) = -\frac{V_E}{k\Omega R} \cos(\theta_E), \quad \frac{\partial^2}{\partial \eta^2} (\tilde{\psi} + \hat{\psi}) = 0 \\ \frac{\partial^2}{\partial \xi \partial \eta} (\tilde{\psi} + \hat{\psi}) = -\frac{V_E}{k\Omega R} \tan(\theta_E) \end{aligned} \quad (12)$$

### 2.3 Specification of radial vorticity

Equation 10 can only be solved if the radial vorticity distribution is specified. Of interest for the utility of the solution is the case where the vorticity attains extreme values (maxima and/or minima) on the boundaries. For the remainder of this paper, the specific case of a *biquartic* (in  $\xi$  and  $\eta$ ) distribution will be investigated, as this distribution allows two extrema in each coordinate direction. Such a

distribution may be expressed in the form

$$\tilde{\omega}_r(\xi, \eta) = \sum_{i=0}^4 \sum_{j=0}^4 \Omega_{ij} \xi^i \eta^j \quad (13)$$

where the  $\Omega_{ij}$  are constants to be specified.

### 2.4 Restriction on longitudinal extent

Although less general than would have been preferable, it is convenient to introduce the following restriction on the longitudinal extent  $\Delta \varphi$  of the solution domain: the solution will only be developed for integral fractions of  $\pi$ , i.e.

$$\Delta \varphi = \pi, \frac{\pi}{2}, \frac{\pi}{3}, \dots$$

or

$$\Delta \varphi^{(k)} = \frac{\pi}{k} \text{ with } k = 1, 2, 3, \dots \quad (14)$$

The superscript  $k$  is implied for the stream function solutions that follow also, but for the sake of clarity is not shown explicitly.

### 3. Stream Function Solution

The stream function solution is now presented for equation 10, incorporating the radial vorticity specification given by equation 13 and restricted to longitudinal extents specified by equation 14, i.e.

$$\begin{aligned} (1 - \xi^2) \frac{\partial}{\partial \xi} \left[ (-\xi^2) \frac{\partial^2}{\partial \xi} (\tilde{\psi} + \hat{\psi}) \right] + k^2 \frac{\partial^2}{\partial \eta^2} (\tilde{\psi} + \hat{\psi}) \\ = (1 - \xi^2) \sum_{i=0}^4 \sum_{j=0}^4 \Omega_{ij} \xi^i \eta^j \end{aligned} \quad (15)$$

subject to the boundary conditions given by equation 11.

#### 3.1 Homogeneous solution

The homogeneous solution of equation 15 is readily obtained by means of the method of separation of variables:

$$\tilde{\psi} = \sum_{m=1}^{\infty} A_m Q_0^{2km}(\xi) \sin(2m\eta) \quad (16)$$

for odd values of  $k$ , and

$$\tilde{\psi} = \sum_{m=1}^{\infty} A_m Q_0^{km}(\xi) \sin(m\eta) \quad (17)$$

for even values of  $k$ . In these expressions, which both satisfy all the boundary conditions with the exception of the one at  $\xi = -\sin(\Delta \theta)$ , the functions  $Q_0^n$  are zero-order associated Legendre functions of the second kind (see appendix A). The constants  $A_m$  have to be determined in conjunction with the remaining boundary condition and the particular solution.

#### 3.2 Particular solution

For the biquartic vorticity distribution under consideration, a valid particular solution is the product of polynomials in  $\xi$  and  $\eta$ . Consider the following form of the particular

integral, which satisfies the same three boundary conditions as the homogenous solution:

$$\hat{\psi} = \xi(1-\xi^2)\eta(\pi-\eta) \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} \xi^i \eta^j \quad (18)$$

Substitution of equation 18, together with equation 16 or 17, as appropriate, into equation 15, yields the following relationship between the  $\Omega_{ij}$ - and  $\Psi_{ij}$ -coefficients:

$$\begin{aligned} & \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} \left[ i(i+1)\xi^{i-1} - 2(i+2)^2 \xi^{i+1} + (i+3)(i+4)\xi^{i+3} \right] \\ & (\pi\eta^{j+1} - \eta^{j+2}) + k^2 \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} \xi^{i+1} \\ & \left[ \pi j(j+1)\eta^{j-1} - (j+1)(j+2)\eta^j \right] = \sum_{i=0}^4 \sum_{j=0}^4 \Omega_{ij} \xi^i \eta^j \end{aligned} \quad (19)$$

Expressions for the  $\Omega_{ij}$ -coefficients in terms of the  $\Psi_{ij}$ -coefficients are obtained by comparing corresponding terms on the left and right hand sides of equation 7. These expressions are given in appendix B.

### 3.3 Determination of homogeneous solution constants

The boundary condition at  $\xi = -\sin(\Delta\theta)$  is utilized to express the  $A_m$ -values in terms of the other parameters; applying this boundary condition to the sum of the homogeneous and particular solutions yields

$$\begin{aligned} & \sum_{m=1}^{\infty} A_m Q_0^{2km} [-\sin(\Delta\theta)] \sin(2m\eta) + \cos^2(\Delta\theta) \\ & \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} [-\sin(\Delta\theta)]^{i+1} (\pi\eta^{j+1} - \eta^{j+2}) = 0 \end{aligned}$$

for odd values of  $k$ , and

$$\begin{aligned} & \sum_{m=1}^{\infty} A_m Q_0^{km} [-\sin(\Delta\theta)] \sin(m\eta) + \cos^2(\Delta\theta) \\ & \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} [-\sin(\Delta\theta)]^{i+1} (\pi\eta^{j+1} - \eta^{j+2}) = 0 \end{aligned}$$

for even values of  $k$ . Expressions for the  $A_m$ -coefficients are obtained by means of a standard Fourier analysis, i.e.

$$\begin{aligned} A_m &= \frac{\cos^2(\Delta\theta)}{Q_0^{2km} [-\sin(\Delta\theta)]} \\ & \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} [-\sin(\Delta\theta)]^{i+1} (I_{m,j+2} - \pi I_{m,j+1}) \end{aligned} \quad (20)$$

for odd values of  $k$ , and

$$\begin{aligned} A_m &= \frac{\cos^2(\Delta\theta)}{Q_0^{km} [-\sin(\Delta\theta)]} \\ & \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} [-\sin(\Delta\theta)]^{i+1} (J_{m,j+2} - \pi J_{m,j+1}) \end{aligned} \quad (21)$$

for even values of  $k$ . Expressions for the definite integrals  $I_{m,n}$  and  $J_{m,n}$  on the right hand side of equations 20 – 21 are given in appendix C.

### 3.4 Determination of particular solution constants

The vortex strength conditions, equation 12, are used to fix the six particular solution constants  $\Psi_{ij}$ . The resulting system of equations are given in appendix D.

## 4. Pressure Field Solution

The pressure gradient is defined from equation 2; expanding this expression and noting that  $\vec{\Omega} = \Omega(\cos\theta\vec{e}_R - \sin\theta\vec{e}_\theta)$ , yields the following expression for the two components of the pressure gradient on the spherical surface:

$$\begin{aligned} \frac{1}{\rho} \frac{\partial p}{\partial \theta} &= -\frac{1}{2} \frac{\partial}{\partial \theta} (V_\theta^2 + V_\varphi^2) + (\omega_r + 2\Omega\cos\theta)V_\varphi \\ \frac{1}{\rho} \frac{\partial p}{\partial \varphi} &= -\frac{1}{2} \frac{\partial}{\partial \varphi} (V_\theta^2 + V_\varphi^2) - (\omega_r + 2\Omega\cos\theta)V_\theta \sin\theta \end{aligned} \quad (22)$$

The pressure distribution is obtained by integrating this expression from a reference pressure at a reference position, first in the latitudinal direction, and then in the longitudinal direction. Using the point  $(r=R, \theta = \frac{\pi}{2}, \varphi = \varphi_0)$  as the reference position, and indicating the reference pressure by  $p_0$ , the result is

$$\frac{p-p_0}{\rho} = -\frac{1}{2}(V_\theta^2 + V_\varphi^2) + 2\Omega\psi\cos\theta + \int_{\varphi_0}^{\varphi} \omega_r \frac{\partial \psi}{\partial \varphi} d\varphi \quad (23)$$

In terms of the transformed coordinates  $\xi$  and  $\eta$ , etc., this expression can be written more explicitly in the dimensionless form

$$\begin{aligned} C_p &= \frac{p-p_0}{\frac{1}{2}\rho R^2 \Omega^2} = -\frac{k^2}{1-\xi^2} \left[ \frac{\partial}{\partial \eta} (\tilde{\psi} + \hat{\psi}) \right]^2 - (1-\xi^2) \\ & \left[ \frac{\partial}{\partial \xi} (\tilde{\psi} + \hat{\psi}) \right]^2 + 2(2\xi + \tilde{\omega}_r)(\tilde{\psi} + \hat{\psi}) - 2 \int_{\varphi_0}^{\eta} \frac{\partial \tilde{\omega}_r}{\partial \eta} (\tilde{\psi} + \hat{\psi}) d\eta \end{aligned} \quad (24)$$

Evaluation of the integral on the right hand side of this expression is simple but tedious; details are given in appendix E.

## 5. Case Study

A case of interest, loosely corresponding to the southern Indian ocean off the south-east coast of South-Africa, is described by the following parameter values:  $\Delta\varphi = 90^\circ$

(i.e.  $k = 2$ ),  $\Delta\theta = 80^\circ$ ,  $\Omega = 72.921 \times 10^{-6}$  rad/s,

$R = 6.378 \times 10^6$  m,  $\theta_W = 40^\circ$ ,  $\theta_E = 60^\circ$ ,

$V_W = 2$  m/s,  $V_E = -0.5$  m/s.

The particular solution coefficient values determined for this parameter set are listed in table 1, and the first six homogeneous solution constants in table 2, showing rapid

convergence of the series solution. A streamline plot is shown in figure 2 and a contour plot of static pressure in figure 3. The western intensification of current strength is clearly visible in the streamline plot. The close correspondence between the direction of the streamlines in figure 2 and the direction of the isobars in figure 3 is also evident. It can therefore be concluded that a qualitatively realistic initial condition set can indeed be generated by means of the analytical solution presented here.

$\psi_{00} = 0.000409$	$\psi_{01} = 0.003789$	$\psi_{02} = -0.001335$
$\psi_{10} = -0.001529$	$\psi_{11} = 0.005205$	$\psi_{12} = -0.001650$

Table 1: Particular solution constants

$A_1 = -0.7256 \times 10^{-6}$	$A_2 = -0.3708 \times 10^{-9}$	$A_3 = -0.7788 \times 10^{-13}$
$A_4 = -0.3232 \times 10^{-17}$	$A_5 = -0.3476 \times 10^{-21}$	$A_6 = -0.7085 \times 10^{-26}$

Table 2: Homogeneous solution constants

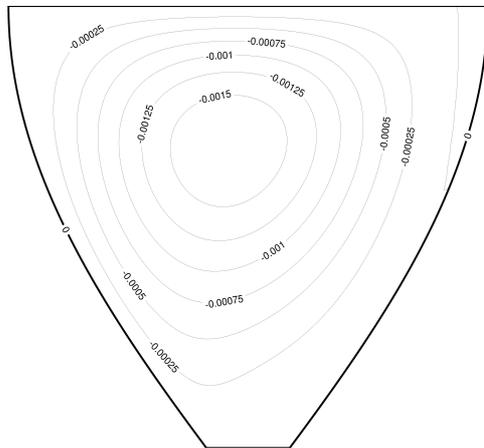


Figure 2: Contours of dimensionless stream function (values decreasing from the center outwards indicate anti-clockwise rotation)

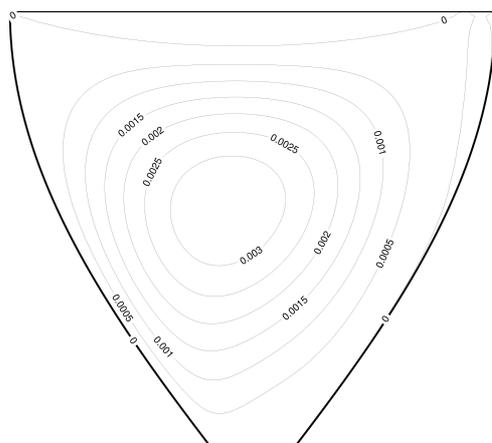


Figure 3: Contours of static pressure coefficient

## 6. Conclusion

An analytical solution for a vortex-type flow of an inviscid fluid on a spherical surface has been presented. The purpose of this analytical solution is that of providing initial conditions to CFD simulations of global ocean circulation,

in order to reduce the CPU time required for such simulations. The seemingly limited applicability of the analytical solution may be extended in situations such as the following:

- ❑ “Ragged” coastlines, i.e. coastlines not aligned with longitudinal and latitudinal coordinate directions: use the initial condition solution in a region within the ocean basin, with zero flow between the boundaries in which the analytical solution is valid and the actual coastline.
- ❑ Ocean basins extending over the equator: patch southern and northern hemispherical solutions together, each with its own  $\Delta\theta$  for latitudinal extent.
- ❑ Ocean basins with longitudinal extent different from any one of the discrete longitudinal extents for which the analytical solution is valid: patch together solutions of different longitudinal extent, possibly overlapping them at common internal boundaries.
- ❑ Ocean basins of finite depth: the solution can be applied for different radii of the spherical surface, thus patching together initial conditions extending into the third (depth) dimension.
- ❑ Ocean basins not extending to the equator: a “small ocean” solution extending to the equator can be subtracted from a “large ocean” solution also extending to the equator (the two solutions must of course share the same longitudinal boundaries).
- ❑ More concentrated currents on the western boundary: more terms can be included in the vorticity distribution, even to the extent of using a full Taylor series expansion of a step function.

## References

- 1 *US Department of the Interior*, Technology White Paper on Ocean Current Energy Potential on the U.S. Outer Continental Shelf, 2006.
- 2 *Miller RN*, Numerical Modelling of Ocean Circulation, Cambridge University Press, 2007.
- 3 *Pond S and Pickard GL*, Introductory Dynamical Geography, Butterworth-Heinemann, Oxford, 1983.
- 4 *Stommel H*, The westward intensification of wind-driven ocean currents, *Transactions, American Geophysical Union*, 1948, 29(2), 202-206.
- 5 *Fofonoff NP*, Steady flow in a frictionless homogeneous ocean, *Journal of Marine Research*, 1954, 13(3), 254-262.
- 6 *Karamcheti K*, Principles of Ideal-Fluid Aerodynamics, Wiley, New York, 1966.
- 7 *Spiegel MR, Lipschutz S and Liu J*, Mathematical Handbook of Formulas and Tables, third edition, McGraw-Hill, New York, 2009.

**Appendix A: Zero-order associated Legendre functions**

The zero-order associated Legendre functions are defined by (see for example Spiegel *et al.*<sup>7</sup>)

$$Q_0^n(x) = \frac{1}{2} (1-x^2)^{n/2} \frac{d^n}{dx^n} \left[ \frac{1+x}{1-x} \right] \quad (A1)$$

Carrying out the indicated differentiations yield the following expressions for the first two functions in the sequence:

$$Q_0^1(x) = \frac{1}{\sqrt{1-x^2}} \quad (A2)$$

$$Q_0^2(x) = \frac{2x}{\sqrt{1-x^2}} \quad (A3)$$

For  $Q_0^3, Q_0^4$  the recurrence formula given by Spiegel *et al.*<sup>7</sup> may be used:

$$Q_0^n = \frac{2(n-1)x}{\sqrt{1-x^2}} Q_0^{n-1} + (n-1)(n-2) Q_0^{n-2} \quad (A4)$$

Using equations 1 and 4 it can be shown that the derivative of the function  $Q_0^n$  is given by

$$Q_0^{n'} = \frac{nx}{1-x^2} Q_0^n + \frac{n(n-1)}{\sqrt{1-x^2}} Q_0^{n-1} \quad (A5)$$

**Appendix B: Vorticity distribution coefficients**

It is convenient to rewrite Equation 19 in the following form:

$$\begin{aligned} \sum_{i=0}^4 \sum_{j=0}^4 \Omega_{ij} \xi^i \eta^j &= \pi \sum_{i=0}^0 \sum_{j=1}^3 \psi_{i+1, j-1} (i+1)(i+2) \xi^i \eta^j \\ &- \sum_{i=0}^0 \sum_{j=2}^4 \psi_{i+1, j-2} (i+1)(i+2) \xi^i \eta^j \\ &- 2\pi \sum_{i=1}^2 \sum_{j=1}^3 \psi_{i-1, j-1} (i+1)^2 \xi^i \eta^j + 2 \sum_{i=1}^2 \sum_{j=2}^4 \psi_{i-1, j-2} (i+1)^2 \xi^i \eta^j \\ &+ \pi \sum_{i=1}^2 \sum_{j=1}^3 \psi_{i-3, j-1} i(i+1) \xi^i \eta^j - \sum_{i=3}^4 \sum_{j=2}^4 \psi_{i-3, j-2} i(i+1) \xi^i \eta^j \\ &+ \pi k^2 \sum_{i=1}^2 \sum_{j=0}^1 \psi_{i-1, j+1} (i+1)(j+2) \xi^i \eta^j \\ &- k^2 \sum_{i=1}^2 \sum_{j=0}^2 \psi_{i-1, j} (j+1)(j+2) \xi^i \eta^j \end{aligned} \quad (B1)$$

Using this expression, it follows that

$$\begin{pmatrix} \Omega_{00} \\ \Omega_{10} \\ \Omega_{01} \\ \Omega_{20} \\ \Omega_{11} \\ \Omega_{02} \\ \Omega_{30} \\ \Omega_{21} \\ \Omega_{12} \\ \Omega_{03} \\ \Omega_{40} \\ \Omega_{31} \\ \Omega_{22} \\ \Omega_{13} \\ \Omega_{04} \\ \Omega_{41} \\ \Omega_{32} \\ \Omega_{23} \\ \Omega_{14} \\ \Omega_{42} \\ \Omega_{33} \\ \Omega_{24} \\ \Omega_{43} \\ \Omega_{34} \\ \Omega_{44} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2k^2 & 0 & 2\pi k^2 & 0 & 0 & 0 \\ 0 & 2\pi & 0 & 0 & 0 & 0 \\ 0 & -2k^2 & 0 & 2\pi k^2 & 0 & 0 \\ -8\pi & 0 & -6k^2 & 0 & 6\pi k^2 & 0 \\ 0 & -2 & 0 & 2\pi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -18\pi & 0 & -6k^2 & 0 & 6\pi k^2 \\ 8 & 0 & -8\pi & 0 & -12k^2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2\pi \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 12\pi & 0 & 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & -18\pi & 0 & -12k^2 \\ 0 & 0 & 8 & 0 & -8\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 20\pi & 0 & 0 & 0 & 0 \\ -12 & 0 & 12\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & 18 & 0 & -18\pi \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & -20 & 0 & 20\pi & 0 & 0 \\ 0 & 0 & -12 & 0 & 12\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & -20 & 0 & 20\pi \\ 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 \end{bmatrix} \begin{pmatrix} \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11} \\ \psi_{02} \\ \psi_{12} \end{pmatrix} \quad (B2)$$

**Appendix C: Integrals for homogeneous solution coefficients**

We need to evaluate integrals of the form

$$I_{m,n} = \frac{4}{\pi} \int_0^{\pi/2} \eta^n \sin(2m\eta) d\eta \text{ for } n = 0, 1, \dots, 4 \quad (C1)$$

and

$$J_{m,n} = \frac{2}{\pi} \int_0^{\pi} \eta^n \sin(m\eta) d\eta \text{ for } n = 0, 1, \dots, 4 \quad (C2)$$

It is easy to show that

$$I_{m,n} = \frac{J_{m,n}}{2^n} \quad (C3)$$

and therefore only the integrals  $J_n, n=0, \dots, 4$ , need to be evaluated; using the relevant integration formulas in chapter 17 of Spiegel *et al.*<sup>7</sup> we obtain the following expressions for these integrals:

$$\begin{aligned}
 J_{m,0} &= \frac{2}{\pi m} (-1)^{m+1} \\
 J_{m,1} &= \frac{2}{m} (-1)^{m+1} = \pi J_{m,0} \\
 J_{m,2} &= \frac{2\pi}{m} (-1)^{m+1} - \frac{4}{\pi m^3} [1 + (-1)^{m+1}] \\
 &= \pi^2 J_{m,0} - \frac{4}{\pi m^3} [1 + (-1)^{m+1}] \\
 J_{m,3} &= \frac{2}{m^3} (-1)^{m+1} (m^2 \pi^2 - 6) = \pi^3 J_{m,0} - \frac{6}{m^2} J_{m,1} \\
 J_{m,4} &= \frac{2\pi}{m^3} (-1)^{m+1} (m^2 \pi^2 - 12) + \frac{48}{\pi m^5} [1 + (-1)^{m+1}] \\
 &= \pi^4 J_{m,0} - \frac{12}{m^2} J_{m,2}
 \end{aligned} \tag{C4}$$

**Appendix D: System of equations for determining the particular solution constants**

For odd values of  $k$ , the system of equations is as follows:

$$\begin{aligned}
 \sum_{m=1}^{\infty} 2m \frac{Q_0^{2km} [-\sin(\theta_W)]}{Q_0^{2km} [-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (I_{m,j+2} - \pi I_{m,j+1}) \Psi_{ij} + \frac{\pi \cos^2(\theta_W)}{\cos^2(\Delta\theta)} \sum_{i=0}^1 [-\sin(\theta_W)]^{i+1} \Psi_{i0} \\
 = -\frac{V_W \cos(\theta_W)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D1}$$

$$\sum_{i=0}^1 [-\sin(\theta_W)]^i (\Psi_{i0} - \pi \Psi_{i1}) = 0 \tag{D2}$$

$$\begin{aligned}
 \sum_{m=1}^{\infty} 2m \frac{Q_0^{2km} [-\sin(\theta_W)]}{Q_0^{2km} [-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (I_{m,j+2} - \pi I_{m,j+1}) \Psi_{ij} + \frac{\pi}{\cos^2(\Delta\theta)} \sum_{i=0}^1 [-\sin(\theta_W)]^i \\
 [(i+2) \cos^2(\theta_W) - 2] \Psi_{i0} = -\frac{V_W \tan(\theta_W)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D3}$$

$$\begin{aligned}
 \sum_{m=1}^{\infty} 2m \frac{Q_0^{2km} [-\sin(\theta_E)]}{Q_0^{2km} [-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (I_{m,j+2} - \pi I_{m,j+1}) \Psi_{ij} - \frac{\cos^2(\theta_E)}{\cos^2(\Delta\theta)} \sum_{i=0}^1 \sum_{j=0}^2 \pi^{j+1} \\
 [-\sin(\theta_E)]^{i+1} \Psi_{ij} = -\frac{V_E \cos(\theta_E)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D4}$$

$$\sum_{i=0}^1 \sum_{j=0}^2 (j+1) \pi^j [-\sin(\theta_E)]^i \Psi_{ij} = 0 \tag{D5}$$

$$\begin{aligned}
 \sum_{m=1}^{\infty} 2m \frac{Q_0^{2km'} [-\sin(\theta_E)]}{Q_0^{2km} [-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (I_{m,j+2} - \pi I_{m,j+1}) \Psi_{ij} - \frac{1}{\cos^2(\Delta\theta)} \sum_{i=0}^1 \sum_{j=0}^2 \pi^{j+1} \\
 [-\sin(\theta_E)]^i [(i+2) \cos^2(\theta_E) - 2] \Psi_{ij} \\
 = -\frac{V_E \tan(\theta_E)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D6}$$

For even values of  $k$ , the system of equations is similar:

$$\begin{aligned}
 \sum_{m=1}^{\infty} m \frac{[-\sin(\theta_W)]}{[-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (J_{m,j+2} - J_{m,j+1}) \Psi_{ij} + \frac{\cos^2(\theta_W)}{\cos^2(\Delta\theta)} \sum_{i=0}^1 [-\sin(\theta_W)]^{i+1} \Psi_{i0} \\
 = -\frac{V_W \cos(\theta_W)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D7}$$

$$\sum_{i=0}^1 [-\sin(\theta_W)]^i (\Psi_{i0} - \pi \Psi_{i1}) = 0 \tag{D8}$$

$$\begin{aligned}
 \sum_{m=1}^{\infty} m \frac{Q_0^{km'} [-\sin(\theta_W)]}{Q_0^{km} [-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (J_{m,j+2} - \pi J_{m,j+1}) \Psi_{ij} + \frac{\pi}{\cos^2(\Delta\theta)} \sum_{i=0}^1 [-\sin(\theta_W)]^i \\
 [(i+2) \cos^2(\theta_W) - 2] \Psi_{i0} = -\frac{V_W \tan(\theta_W)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D9}$$

$$\begin{aligned}
 \sum_{m=1}^{\infty} m (-1)^m \frac{Q_0^{km} [-\sin(\theta_W)]}{Q_0^{km} [-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (J_{m,j+2} - \pi J_{m,j+1}) \Psi_{ij} - \frac{\cos^2(\theta_E)}{\cos^2(\Delta\theta)} \sum_{i=0}^1 \sum_{j=0}^2 \pi^{j+1} \\
 [-\sin(\theta_E)]^{i+1} \Psi_{ij} = -\frac{V_E \cos(\theta_E)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D10}$$

$$\sum_{i=0}^1 \sum_{j=0}^2 (j+1) \pi^j [-\sin(\theta_E)]^i \Psi_{ij} = 0 \tag{D11}$$

$$\begin{aligned}
 \sum_{m=1}^{\infty} m (-1)^m \frac{Q_0^{km'} [-\sin(\theta_E)]}{Q_0^{km} [-\sin(\Delta\theta)]} \sum_{i=0}^1 \sum_{j=0}^2 [-\sin(\Delta\theta)]^{i+1} \\
 (J_{m,j+2} - \pi J_{m,j+1}) \Psi_{ij} \\
 - \frac{1}{\cos^2(\Delta\theta)} \sum_{i=0}^1 \sum_{j=0}^2 \pi^{j+1} [-\sin(\theta_E)]^i \\
 [(i+2) \cos^2(\theta_E) - 2] \Psi_{ij} = -\frac{V_E \tan(\theta_E)}{k\Omega R \cos^2(\Delta\theta)}
 \end{aligned} \tag{D12}$$

**Appendix E: Integrals for determination of pressure field**

We need to evaluate the integral

$$\int_0^\eta \frac{\partial \tilde{\omega}_r}{\partial \eta} (\tilde{\psi} + \hat{\psi}) d\eta = \sum_{\ell=0}^4 \sum_{n=1}^4 n \Omega_{\ell n} \xi^\ell \left[ \sum_{m=1}^{\infty} A_m Q_0^{2km}(\xi) \int_0^\eta \eta^{n-1} \sin(2m\eta) d\eta + \xi(1-\xi^2) \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} \xi^i \int_0^\eta (\pi \eta^{j+n} - \eta^{j+n+1}) d\eta \right] \quad (E1)$$

for odd values of  $k$ , and

$$\int_0^\eta \frac{\partial \tilde{\omega}_r}{\partial \eta} (\tilde{\psi} + \hat{\psi}) d\eta = \sum_{\ell=0}^4 \sum_{n=1}^4 n \Omega_{\ell n} \xi^\ell \left[ \sum_{m=1}^{\infty} A_m Q_0^{km}(\xi) \int_0^\eta \eta^{n-1} \sin(m\eta) d\eta + \xi(1-\xi^2) \sum_{i=0}^1 \sum_{j=0}^2 \Psi_{ij} \xi^i \int_0^\eta (\pi \eta^{j+n} - \eta^{j+n+1}) d\eta \right] \quad (E2)$$

for even values of  $k$ . All the integrals in these two expressions can be evaluated analytically; see for example chapter 17 in Spiegel *et al.*<sup>7</sup>