Art. #1682, 12 pages, https://doi.org/10.15700/saje.v40n2a1682

The integration of semiotic resources and modalities in the teaching of geometry in a Grade 9 class in a South African high school: The four cases of congruency

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In this article we examine the nature of inter-semiotic and intermodal construction in the exposition of a solution for a geometry rider. In the tradition of hermeneutic phenomenology, this case study involved an exploration of the oral discourse and visual texts used in a mathematics lesson. This research was intended to contribute to the understanding of the difficulties in teaching and learning geometry at school level. Results indicate that relational markings, oral and visual modalities in conjunction with gesturing constitute the primary semiotic resources employed by the teacher. This leads to the conclusion that the semiotic perspective, in conjunction with other perspectives on geometry teaching in schools, may provide a mechanism by which to reflect on the complexity of geometry teaching and learning in schools.

Keywords: argumentation; congruency; diagrams; geometry; geometry riders; logical; relational markings; semiotics; spatial reasoning; teaching and learning; visual texts

Introduction

The literature indicates that many learners and teachers have difficulties in spatial reasoning (Cheah, Herbst, Ludwig, Richard & Scaglia, 2017; Elia & Gagatsis, 2003; Marchis, 2012). For this reason, such difficulties experienced in school geometry form a focus area within the broader research literature on the teaching and learning of geometry. This research was intended to contribute to the understanding of the difficulties in teaching and learning geometry at school level.

Literature Review

In most countries across the world the goals of geometry in the school curricula are considered to be the development of the skills of visualisation, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argumentation and the capacity to produce proof (Horsman, 2019; Jones & Tzekaki, 2016; Kuzniak, 2018).

> In South Africa, these aims are captured in the curriculum for the Senior Phase (Grades 7–9) as follows:
> The study of Space and Shape improves understanding and appreciation of the pattern, precision, achievement and beauty in natural and cultural forms. It focuses on the properties, relationships, orientations, positions and transformations of two-dimensional shapes and three-dimensional objects (Department of Basic Education (DBE), Republic of South Africa, 2011a:10).

The Further Education and Training Band (Grades 10–12) invokes the same directive in its specific aims for mathematics: "use spatial skills and properties of shapes and objects to identify, pose and solve problems creatively and critically" (DBE, Republic of South Africa, 2011b:9).

What emerges from these goals is that they still reflect the four goals formulated in 1984 by the National Council of Teachers of Mathematics (NCTM). These goals were central to the argument for the inclusion of geometry in school curricula: "(1) to develop logical thinking abilities; (2) to develop spatial intuition about the real world; (3) to impart the knowledge needed for further study in mathematics; and (4) to teach the reading and interpretation of mathematical arguments" (Suydam, 1985:481).

Despite these universally accepted goals, the inclusion of geometry in school mathematics curricula has always been a controversial issue. More than thirty years ago the United Nations Educational, Scientific and Cultural Organization (UNESCO) addressed this matter on the basis that "[t]here is no consensus on the content of the school geometry curriculum" (Morris, 1986:i). Regardless of the noble goals for geometry education in schools, Fey (1984:31) concurs by stating that geometry seems to be "the most troubled and controversial topic in school mathematics today."

Recent literature on the teaching and learning of geometry (Horsman, 2019:99) opine that multiple studies report that "the effective teaching and learning of proof still eludes us. Even those who are successful in achieving high results seem to have to a certain extent, merely rote learn two-column proof arguments and are challenged when faced with non-routine geometry proof tasks." Jojo (2017:246) confirms this state of affairs for the South African context: "… despite various efforts invested in professional development of mathematics teachers, there appears to be very little change towards learning environments conducive to geometry teaching. Consequently, performance in mathematics continues to be poor."

In the first round of the revision of school curricula in South Africa (Revised National Curriculum Statement [RNCS]), geometry was excluded from the Grade 10 to 12 mathematics curricula. However, it was included in the advanced mathematics curriculum which was followed by a select few learners only. This exclusion of geometry for most mathematics learners in Grades 10 to 12 was based on the perception that it was difficult to teach. This was ascribed to teacher readiness (or lack thereof) to deal with the instruction of school geometry. For the current curriculum (Curriculum and Assessment Policy Statement [CAPS]), geometry has once again been included in the Grade 10 to 12 mathematics curriculum.

In their synthesis of research reports (spanning a ten-year period) from the proceedings of the annual conferences of the Psychology of Mathematics, Jones and Tzekaki (2016:109) assert that:

> the emphasis of subsequent geometry education research has increasingly been on the use of technology (especially forms of dynamic geometry software) and how this impacts on geometry teaching and learners' geometrical thinking (especially on the teaching and learning of geometrical reasoning and proving), on teachers' geometric content knowledge, and on teacher development for geometry education.

Despite the rich corpus of research in geometry teaching and learning, there is widespread agreement that learners' achievement in geometry in high-stakes examinations is at unsatisfactory levels (Adeniji, Ameen, Dambatta & Orilonise, 2018; Renne, 2004; Zakariyya, Ndagara & Yahaya, 2016). Clements and Battista (1992:422) describe achievement in school geometry as presenting a "depressing picture of students' knowledge of geometry [and] students' misconceptions." Atebe and Schäfer (2009) state that the teaching and learning of geometry is one of the most disappointing experiences in many schools across nations. Similar sentiments are expressed by other researchers (Giannakopoulos, 2017; Herbst, 2006; Sinclair & Moss, 2012).

If performance in the high-stakes "National Senior Certificate (NSC): Mathematics for geometry in South Africa" is taken into account, then it attests to this picture. Figure 1 gives a glimpse of performance in geometry over the last three years in relation to other topics examined in the second paper of the NSC Mathematics Examination. It is clear that in this period attainment in the geometry and the closely related trigonometry exams did not exceed an average of 40%.

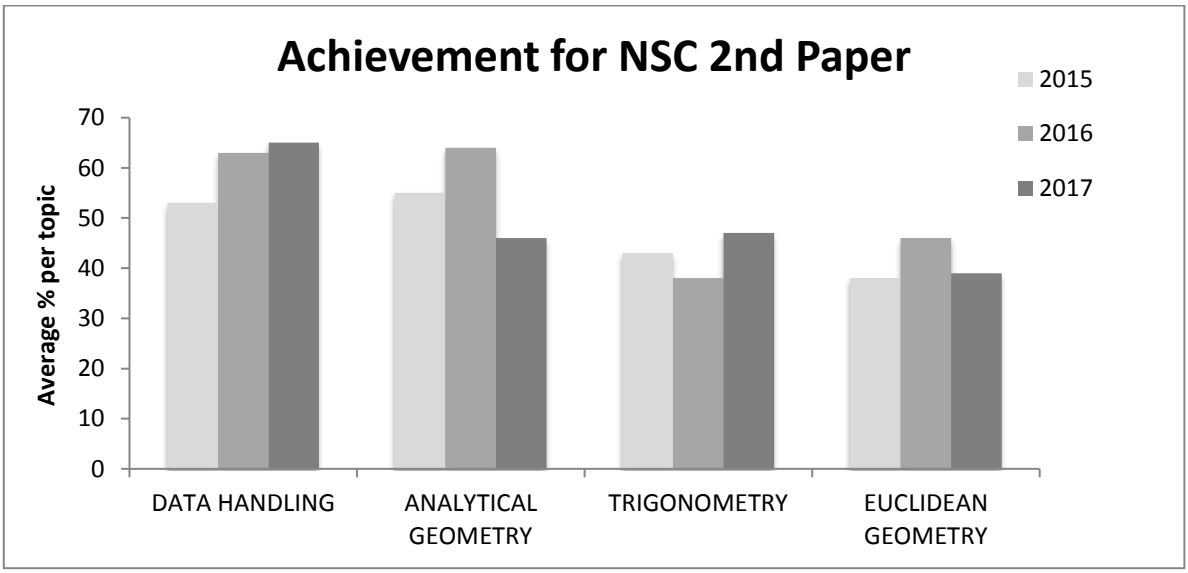


Figure 1 Performance in geometry 2015–2017 (DBE, 2018)

It is evident from the foregoing review of literature that teachers and learners in high schools find geometry extremely challenging. One of the reasons for this is that geometry generally lacks the algorithmic structure that exists in algebra. For example, in algebra, when a quadratic equation is to be solved, there are set routines for transforming the given equation into the standard form, thus making it possible to use a formula. Reasons are not required for justifying the calculations involved in each step. In geometry however, it is different. There are generally no set routines, and each step has to be justified with a reason which appeals to some definition or theorem. This feature of geometry implies that there is reliance on various mathematical objects such as definitions, theorems, or diagrams when engaging in solving geometry riders. This engagement proves to be epistemologically complex.

In order to develop a deeper understanding of the epistemological complexity in the teaching and learning of geometry, we cast a semiotic gaze on the teaching behaviour in one classroom in South Africa

where geometry was being taught to a Grade 9 class. In this article we explore the nature of the inter-semiotic and intermodal construction of meaning which take place in a mathematics classroom in the context of a geometry lesson. We focus on the following aspects of semiotic resources: (a) the function and use of a specific semiotic resource; (b) the aim of a specific semiotic resource; (c) how the semiotic resource features in a teaching sequence; and (d) the challenges that may be expected when a specific semiotic resource is deployed in the instructional sequence.

Conceptual Framework

Semiotic perspectives in the teaching and learning of mathematics have become an important asset to researchers because of the explanatory qualities that such perspectives afford (Duval, 2017). According to Sáenz-Ludlow and Presmeg (2006) this affordance is a consequence of the iconicity and indexicality embedded in mathematical objects, which in this article are congruent and non-congruent triangles.

Semiotic theories deserve attention here because they allow for new perspectives on knowing and knowledge, on representing and representation, on communicating and communication, and on teaching and learning. Such insights are useful for understanding the relationships that encompass the meaning-making process of individuals in sociocultural and cognitively challenging contexts (Sáenz-Ludlow & Kadunz, 2016; Sáenz-Ludlow & Presmeg, 2006).

Ernest (2006:67) provides further justification for the use of semiotic theories in mathematics education.

> Mathematics is an area of human endeavour and knowledge that is known above all ... for its unique range of signs and sign-based activity. So it seems appropriate to apply the science of signs to mathematics. Likewise in schooling, learners meet a whole new range of signs and symbolising functions in mathematics. So again it seems appropriate to adopt a sign-orientated perspective from which to examine school mathematics.

Semiotics is the theory that explains the production and interpretation of meaning. It is thus the study of how people construct meaning in both verbal and non-verbal ways (Duval, 2017). As such, semiotics is an important construct in understanding classroom discourses. Semiotics deals with the study of signs and symbols, which can be both discursive and non-discursive. For the purpose of this article discursive symbolism is language-based thought and meaning, while non-discursive symbolism is non-verbal emotion and meaning as found in art, music, dance and so on.

There are two major traditions in the semiotic literature namely, the ideas of De Saussure (1857–1913) and the ideas propagated by Peirce (1839–1914) (Chandler, 2017). De Saussure proposes a sign as a dyadic structure comprising a signifier and a signified. Peirce extends this by proposing a triadic structure composed of the object, the "representamen" and the "interpretant." The main difference between the two streams is that for Peirce (a philosopher and mathematician) the sign is attached to something concrete while for De Saussure (a linguist) the sign embodies an abstraction of the concrete object. Hence for De Saussure signification requires only two constructs, that of a signifier and the signified.

De Saussure's linguistic focus on semiotics revolves around engendering and processing signs and making them meaningful. For Peirce the act of making meaning of the relation between the signifier and the signified, which he calls the "interpretant," appeals to the teaching and learning of mathematics. This is an important insight for those working in mathematics education.

To understand meaning making is also to understand the active role of the interpreting person in the re-construction of the real object of a sign from the cues and hints carried out by sign-vehicles, which indicate only certain aspects of the real object. For example, in mathematics, when we use the symbol x to signify a variable or an unknown number, the interpretation that this symbol is a number and not an alphabetic character gives meaning to the construct of a polynomial such as $2x^3 - 3x^2 - 4$. This act of assigning meaning is what Peirce calls the "interpretant." It is also this aspect that serves as a basis for the learning of geometry where diagrams are used to represent abstract concepts such as points, lines and polygons in general (Sáenz-Ludlow & Presmeg, 2006:3).

This triadic relationship between an object, "representamen" and "interpretant" explains some of the complexities related to the study of Geometry. For instance, when it is stated that *Figure ABCD is a parallelogram* and one of the following diagrams (Figure 2) is constructed, then the conception of a sign plays itself out. The related complexities for the "interpretant" will then be displayed by the signs signifying different concepts (equality for the diagram on the left-hand side and parallelism for the right-hand side).

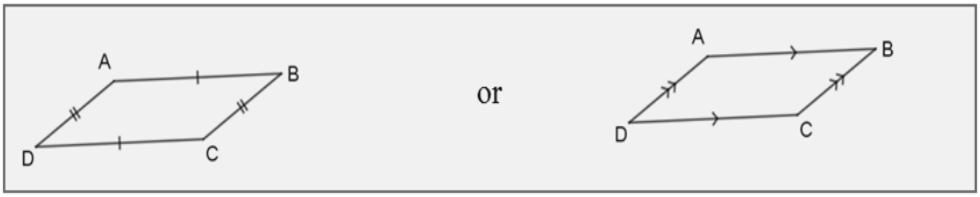
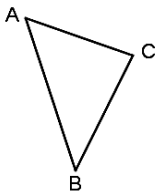


Figure 2 Two possible representations of the symbol *parallelogram ABCD*

Hence, what is important is Peirce's conceptualisation of semiotics, which is more than merely gaining information from signs or making sense of them. "Peircean semiotics implies sign *mediation*; it is deeper and more comprehensive than the ordinary expressions 'derivation of meaning' or 'interpretation'" (Merrel, n.d.:para. 31).

For the purposes of this article the distinction between the semiotics of De Saussure and those of Peirce will not be pursued further. Instead, the focus falls on the idea of signs in the Peircean tradition, which allows us to identify different kinds of signs. This facility, according to Otte (2006), forms one of the important achievements in Peircean semiotics. In general, there is agreement on the existence of three different kinds of signs, namely, icons, indices and symbols. These different types of signs are illustrated in Table 1, which outlines strategies for exploring the concept of a triangle.

Table 1 A geometric illustration of the three types of symbols

Symbol	Type of symbol	Explanation
	This is an iconic sign	It stands for a triangle by resembling it. Learners identify the sign because it resembles a specific type of shape.
Triangle ABC	This is an indexical sign	It is causally related to the object via the sound of the utterance. Learners hear the word and it conjures up a picture of the shape in their minds.
ΔABC	This is a symbolic sign	It is a cultural convention to be acquired by the learner.

Sáenz-Ludlow and Presmeg (2006:8) make an important point on how semiotics plays itself out in the teaching and learning of mathematics:

> To communicate mathematically in the classroom, the teacher has to have the flexibility to move within and between different semiotic systems (ordinary language, mathematical sublanguage, mathematical notations, diagrams, graphs, gestures, etc.) in order to *refer* to mathematical objects that are other than concrete, and to *address* the students by means of material signifiers in order to *express* the teacher's interpretation and contextualization of mathematical objects.

The foregoing narrative illuminates certain constructs that pertain to the teaching and learning of the mathematics of congruency. For the purpose of framing this discussion these have been extrapolated and are presented in the sub-section which follows.

1) *Semiotic resources* are a means to facilitating meaning making in the teaching and learning of mathematics. They are actions, materials and artefacts used for communicative purposes in the classroom. Examples of such semiotic resources are:
   - mathematical language used in introducing new knowledge or explaining concepts and procedures;
   - mathematical symbols embedded in mathematical processes, procedures and relations so that they are amenable to transformations; and
   - mathematical diagrams used when linking linguistic descriptions to symbols in order to solve mathematical problems.

The inter-semiotic relations between the three semiotic resources constitute an important foundation for the construction of mathematical knowledge (O'Halloran, 2011). Hence there are semantic transformations which occur during the teaching and learning of mathematics. Within the classroom discourse there may be a shift from language (to introduce a concept or a problem) to a diagrammatic rendition (to represent the relations between the mathematical components) to mathematical symbolism (to capture the relations between these mathematical components to solve the problem). Such transformation may be conceptualised as a semantic circuit or a semantic move.

The idea of a *move* was introduced by Cooney, Davis and Henderson (1975:92) to describe patterns of

didactical actions used to teach or explain particular constructs in school mathematics.

2) *Relational markings* are the symbols that convey geometric properties when they are applied to parts in the geometrical diagram. Examples of relational markings include the small squares that indicate right angles, small arcs that indicate equal angles, hash marks that indicate equal line segments, and sets of arrows that indicate sets of parallel lines. These markings are semiotic resources through which the diagram directly communicates geometric properties without any supporting literal or symbolic statements.
3) *Modalities* refer to the way in which semiotic resources are practically employed in the process of teaching and learning. In the teaching of mathematics, the following list of features serves to illustrate ways in which semiotic resources are employed in the classroom.
    - Oral: describes the modality of employing a semiotic resource to transmit information by word of mouth.
    - Visual: describes the act of transmitting information through the sense of seeing.
    - Haptic: describes the modality which involves employing a semiotic resource using the act of touching or manipulating concrete objects.
4) *The semantic hyperspace* is what arises from the integration of semiotic resources across modalities.
5) *A semiotic node* represents the nexus between two or more semiotic resources.

Semiotics in the teaching of geometry

Dimmel and Herbst (2015:147) contend that geometrical diagrams use the visual features of specific drawn objects to convey meaning about generic mathematical entities. This is corroborated by Otte (2006:15) who contends that "[m]athematics is essentially diagrammatical thinking. Diagrams and diagrammatoidal figures are intended to be applied towards the better understanding of states of things, whether experienced, or read of, or imagined" (Dimmel & Herbst, 2015:147).

> Similarly, Mudaly (2012:30) concludes that
> [d]iagrams can be effective tools for sense making and should be used wisely when presenting word problems to students. Self-explanatory diagrams are true mediating artefacts that help learners develop a better understanding of the mathematical problem; hence, constituting a possible means to solve the problem.

This idea can be seen to acquire further theoretical explication in Fischbein's (1993:140–149) assertion that geometry diagrams are essentially dualistic: on the one hand, they are objects that display spatio-graphical characteristics, while on the other hand, they are signs that represent general concepts which have theoretical properties. Students are challenged by this duality and often teachers are not aware that it poses a challenge when learners read too much into the spatio-graphical features of a geometry diagram (Presmeg, Radford, Roth & Kadunz, 2018a, 2018b; Sáenz-Ludlow & Kadunz, 2016).

Duval (2006:107) identifies this dualism as the element which gives rise to the difficulties in learning mathematics. He asserts that difficulties arise when students have to work with different semiotic systems, "and that manifests itself in the fact that the ability to change from one representation system to another is very often the critical threshold for progress in learning and for problem solving."

Thus, a problem in the teaching of geometry in particular is that there is an underestimation of the cognitive complexity involved in translating between different semiotic systems. Duval (2000:1-55–1-61) draws attention to this complexity by stating that "this association is cognitively complex because in most cases it goes against the common association between words and shapes and because its use runs against the perceptual obviousness." Iori (2017:286) highlights the fact that translating from semiotic register A into semiotic register B may constitute a different cognitive task than translating in the reverse direction. This explains the fact that it is easier to draw a graph from a given equation, than to derive the equation from its graph.

The research reported on in this article used some of the above constructs to pursue the question on how semiotics is employed in an actual mathematics classroom. In particular, the nature of inter-semiotic and intermodal expansion of meanings which takes place in a mathematics classroom in the context of a geometry lesson is pursued.

Methodology, Data Gathering and Analysis Procedures

This case study involved an exploration of the oral discourse and visual texts used in a mathematics lesson. The study concerned a teacher teaching geometry to a Grade 9 class. The teacher was a participant in a continuing professional development project, the Local Evidence-Driven Improvement of Mathematics Teaching and Learning Initiative (LEDIMTALI), focussing on high quality teaching to enhance achievement in high-stakes mathematics examinations (Julie, 2016). As was the case for other schools participating in the project, the school in question serves learners from low socio-economic backgrounds. The Grade 9 class had 32 learners and the language of instruction was Afrikaans. The lesson was part of the teacher's normal teaching plan for the quarter as prescribed in the pace setters in the CAPS (DBE, Republic of South Africa, 2011b).

The research design is based on the qualitative research tradition employing hermeneutic phenomenology as the research methodology. Using participant observation, based on the analysis of a video recording of a lesson on congruency, the data for analysis was obtained (Marczyk, De Matteo & Festinger, 2005; Taylor, Bogdan & DeVault, 2016).

With permission from the teacher, the entire lesson dealing with congruency was video-recorded. Adhering to the ethical principle of anonymity, the facial images extracted from the recording, have been deleted to avoid identification of either the

teacher or the learners. The video-recording focused on the teacher's actions and interactions. The video-recording was subjected to normal qualitative data analysis. A confirmatory stance was taken since the notions of the semiotic interactions mentioned above drove the analysis.

The first author collected the data. He did a first round of analysis of the data essentially searching for occurrences of the semiotic constructs. The data and his analyses were distributed to the other two authors. They independently checked the outcome of the first analysis raising issues and disagreements with the first author's interpretation. This iterative process of competitive argumentation was followed until consensus was reached regarding the interpretation of data segments and its fit or not to the semiotic constructs. To further strengthen the analysis, a preliminary version of the results was presented to an extended group of mathematics educators participating in the LEDIMTALI project. Relevant comments were incorporated in a pilot version of the article delivered at a conference by the first author and also attended by the third author. The two captured the issues raised by conference participants, and all three authors subsequently discussed these. The results that follow are the outcome of the entire analysis process.

Findings

In this section of the article the instructional sequence for the teaching episode is described and the inter-semiotic and intermodal "expansion of meaning-making" is extracted.

The lesson starts with a revision of the four cases of congruency. The teacher introduces the lesson by saying:

> T: *OK, daar is 4 voorwaardes van kongruensie* (Okay, there are four conditions for congruency).

The teacher then proceeds to introduce the first case of congruency. She begins by drawing two triangles on the board, marking the sides that are equal in both. The result is illustrated in Figure 3 below.

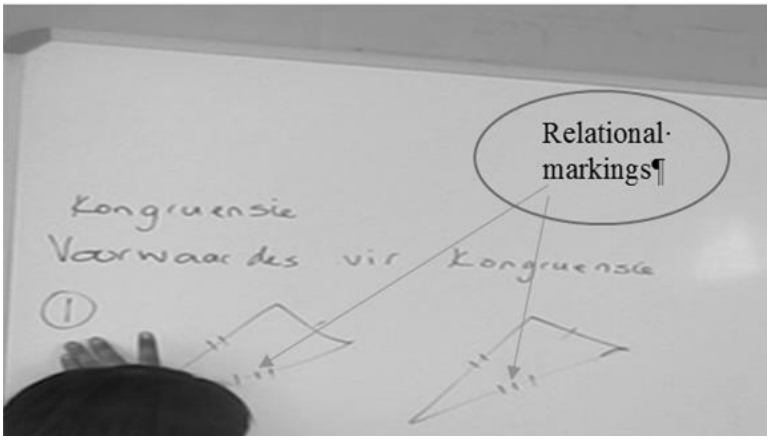


Figure 3 Examples of relational markings

She then proceeds as follows:

> T: *OK, hier is die eerste geval van kongruensie. Dis waar die drie sye van die een driehoek gelyk is aan die drie sye van die ander driehoek* (Okay, here is the first case of congruency. It is where three sides of one triangle are equal to three sides of the other triangle).

The equalities are indicated by relational markings as shown in Figure 1.

She then moves to the board and points at one side of the triangle, and then at the corresponding equal side of the other triangle while explaining as follows:

> T: *In die eerste geval kry jy dat gegee word dat alle sye wat gelyk is aan mekaar ooreenkomstig gemerk word*. (In the first case you find that all the equal sides are marked correspondingly.) *So hierdie sy* (So this side) – indexing a specific side in the first triangle – *is gelyk aan daardie sy* (is equal to that side) – indexing the corresponding equal side in the second triangle.

What is observed is that the teacher almost simultaneously transitions between the linguistic and the diagrammatic. This transitioning is facilitated through the use of indexical gestures. The semiotic node – the nexus between two or more semiotic resources – clearly provides the learners with a way to better interpret the mathematical diagram and to understand what it communicates.

In explaining the example, the teacher spends most of the time writing on the board. This is perhaps one of the most notable features of mathematics lessons. In this regard there seems to be an inter-semiotic relationship, since the "writing" and "talking" parts of teaching are mutually elaborative. Writing and talking are expositions where instances of semiotic nodes underpin the crucial aspect of mediating an understanding of geometrical concepts and geometrical reasoning. As the symbols and diagrams are on the board, the teacher can point to these semiotic resources directly. The teacher employs

these indexical gestures to facilitate an elucidation of the four cases of congruency.

The illustrative example involves the teacher actually doing the mathematics. Having outlined the solution strategy, the teacher works through the details of the proof in a Socratic fashion. The teacher indexes the references to the three elements of the proof of congruency by quickly pointing to the three statements.

The teacher then summarises the procedures before assigning an example for the learners to do on their own. The teacher initiates doing the example by explaining the solution strategy in broad terms. This is referred to as formulating the key idea before giving the written formulation. In this way the teacher employs a typical pedagogical orientation device (First tell them what you are going to do; then do it). In doing so the teacher assists the learners to make sense of the sequence of steps required to prove one triangle congruent to another.

The teacher wants to ensure that the learners understand why things are done in a particular way. To this end the teacher employs certain semiotic moves. A semiotic move comes about through the integration of more than one semiotic resource coupled with certain modalities of mediation. This is similar to the setting up of a semantic hyperspace. The semantic hyperspace is further structured by scaffolding. Using language and a diagram, mathematical symbolism as indicated by the written line, *In ΔABC en ΔBDF is*, is introduced, as indicated in Figure 4. Further scaffolding is provided through the use of the numbers 1, 2, 3 as indexical signs to indicate that there are three statements to be constructed in the proof, as illustrated in Figure 4.

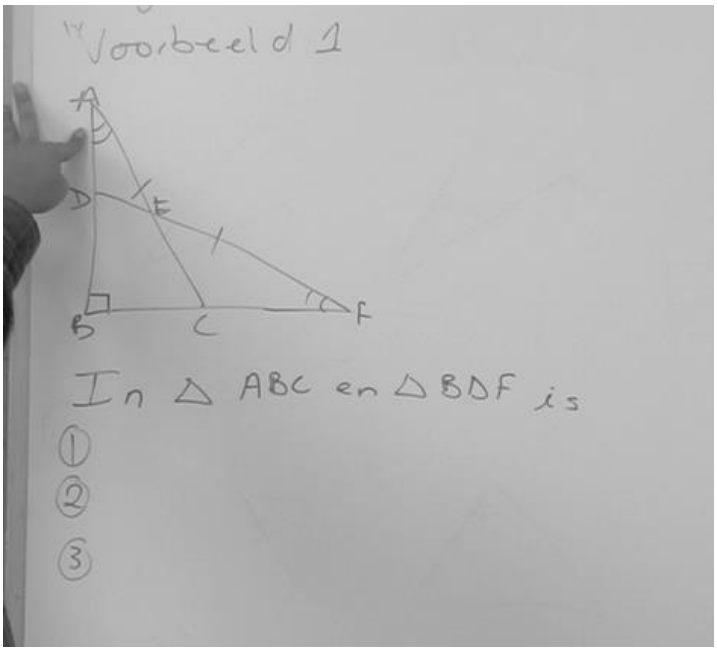


Figure 4 Teacher scaffolding learners' ideas

The discourse unfolds as follows:

T: *Hier het ons twee driehoeke* (Here we have two triangles).

With this utterance the names of the two triangles are written down and the sides that make up the respective triangles are pointed out. This is another demonstration of the employment of a semiotic node. At this point language is used together with a diagram (an iconic sign) and an indexical gesture to focus the learners' attention on the salient features of the task at hand. Radford, Demers, Guzmán and Cerulli (2003:59) identify semiotic nodes using a similar process of conceptualisation: "Along with gestures, the teacher uses locative words and time-bound expressions to achieve a coordination of time, space, and movement. This is an example of a semiotic node."

T: *Daar is ook inligting wat gegee is* (Information is also provided).

This statement by the teacher points to the relational markings on the diagram which show two sides given that are equal.

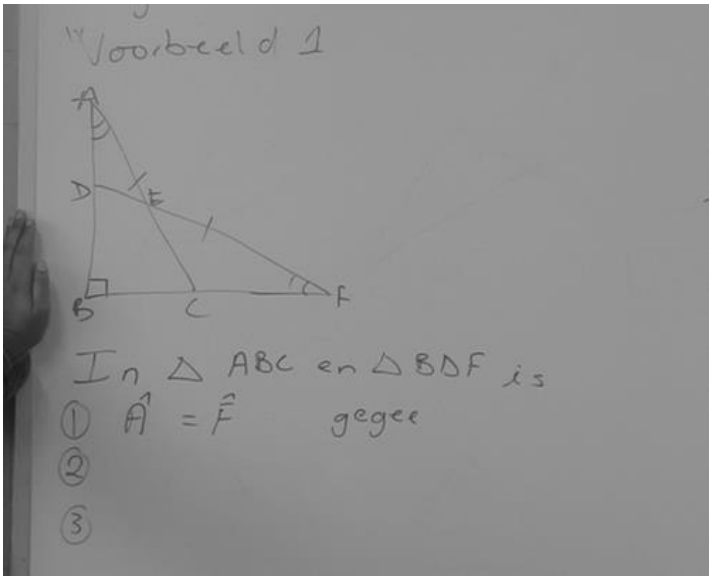


Figure 5 Illustration of how the first statement in the proof is constructed

Figure 5 illustrates how the teacher integrates the use of mathematical symbolism: $\hat{A} = \hat{F}$ *gegee* (given) to structure the solution. Written words, symbolic signs and the spoken word are used to mediate the first step of the solution. Notably, the two triangles are not labelled with attention to the sequence normally accepted for writing the usual order of the vertices in this type of solution strategy. Furthermore, one given or stated condition, $\hat{A} = \hat{F}$, is written down next to the indexical sign, ① . This is the number 1 circled in Figure 5. The other two steps are left open for learners to complete.

The teacher then outlines the solution strategy. This will be referred to as prospective mediation. Prospective mediation alerts the learners to a soon-to-be-followed procedure or process as is evident in the following injunction:

> T: *Jy moet vir my bewys watter hoeke of watter sye in die twee driehoeke is gelyk aan mekaar. Jy moet vir my ten minste drie soortgelyke elemente, sye of hoeke gelyk bewys* (You must prove to me which angles or which sides in the two triangles are equal to each other. You must show that at least three similar elements, sides or angles equal).

Thus, in mediating prospectively the teacher scaffolds the learners' thinking processes. However, the teacher does not provide the actual conditions of congruence but alludes to the fact that there should be three elements which may include only sides, or sides and an angle, or angles and a side in a particular configuration. The generality of the teacher's statement is ostensibly to draw the learners' attention towards applying their knowledge of the cases for congruency. Figure 6 illustrates how the teacher uses semiotic resources at her disposal, in which case, the use of an indexical gesture may be observed emphasising the need for three statements to be constructed.

This reinforces the indexical symbols, 1, 2 and 3, given in Figure 4, which are intended to direct the solution path. Figure 6 instantiates an example of a semiotic node where the teacher uses three different semiotic resources to stress the fact that they have now reached a critical juncture on the solution path. At this point the stage has been set for drawing an informed conclusion.

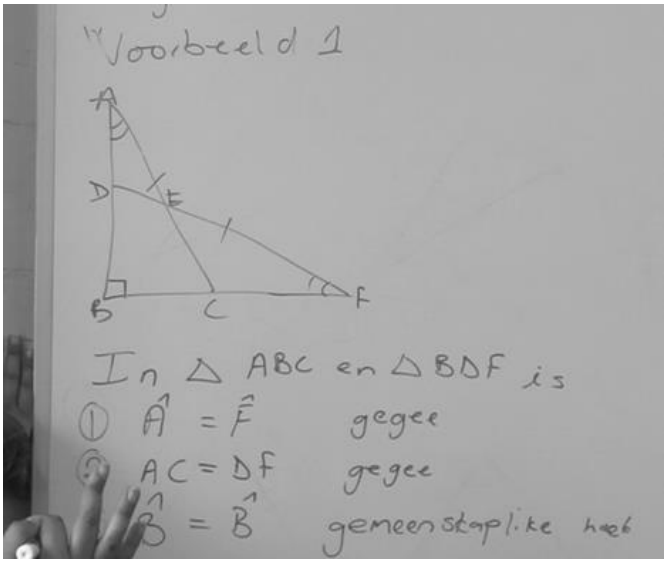


Figure 6 An instance of a semiotic node in the review of the solution path

After reaching a stage where it has been proven that three elements of the one triangle are equal to three corresponding elements of the second triangle, the teacher realises that another critical stage in the solution is at hand, and that this stage requires careful negotiation. The first strategy she deploys is to refer to the introductory part of her lesson. She reminds the learners of the four cases of congruency by repeating what was said in the introduction:

> T: *OK, daar is 4 voorwaardes van kongruensie* (OK, there are four conditions for congruency).

She then poses the question to the learners:

> T: *Watter geval van kongruensie het ons hier bewys?* (Which case of congruency did we prove here?)

This is an important question as it is intended to make the connection between the three statements constructed in the solution procedure and the four cases of congruency. Learners are instructed to consider the cases where we have two angles and the corresponding side as the case of congruency, although in the written account there is an angle, a side and an included angle. An interesting didactical move on the part of the teacher may be observed when she writes the triangle vertices in a particular order (the Δ sign was inadvertently omitted when naming the second triangle).

Figure 7 illustrates the conclusion of the proof. In this conclusion the teacher emphasises an important strategy for denoting the two congruent triangles.

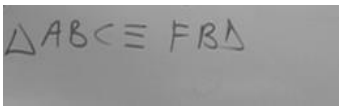


Figure 7 The teacher aligns the two triangles using the equal angles which represent the vertices

This strategy is important as it ensures that the equal elements of the two triangles are written in the same order. In this way the teacher leads the learners to observe that the correct case of congruency that applies in the case of this example is HHS (AAS – Angle, Angle, Side), as illustrated in Figure 8.

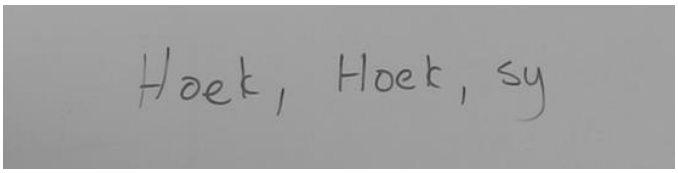


Figure 8 The teacher writes the correct case of congruence using words as indexical signs

Discussion

In this article we analysed how teachers used multisemiotic resources to assist learners to make sense of the concept of congruency and apply this in solving a geometric rider. Although language is the most significant mode of teaching in order to facilitate learning, meanings are made, distributed, received, interpreted and remade through many representational and communicative modes – not just through oral or written language (Moro, Mortimer & Tiberghien, 2019).

The results of this research indicate the interplay between inter-semiotic and intermodal construction in the quest to teach the intricacies of de-

veloping a proof in an expository way. The integration of semiotic resources in the classroom takes place in an epistemologically "togethering" space. This epistemologically "togethering" space is a semantic hyperspace (O'Halloran, 2011:218). It is especially pertinent in the teaching of geometry where geometrical diagrams are used to convey meaning about generic mathematical entities. Coupled with the various semiotic modalities employed, the semantic hyperspace is in continual flux due to resemiotization (O'Halloran, 2011:218). Research on multisemiotics (semiotic hyperspace) contends that participants in interactive contexts (e.g., classrooms) use multiple semiotic resources and language, symbols, images, and embodied actions to make sense of and communicate ideas (Martínez & Dominguez, 2018:3).

The analysis of the foregoing teaching episode illustrates the occurrence of multisemiotics manifesting this particular classroom, since discourse transitioning takes place between different semiotic modalities. This underscores the notion that mathematics teaching, and the learning of mathematics are essentially symbolic practices in which signs are invented, used, or recreated to facilitate cognitive operations or purposes.

In relation to the research investigation discussed here, the results illuminate "the cognitive import of gestures, words, and artefacts in the production of graphical as well as algebraic symbolic expressions" (Radford et al., 2003:55). Accordingly, we recognise geometric concepts multisemiotic constructs that are simultaneously verbal, mathematical, visual-graphical and actional-operational. Thus, multimodality seems to be a crucial feature to consider when studying the teaching and learning of mathematics (Moro et al., 2019:2).

Conclusion

The relevance of semiotics as a tool for understanding and describing teaching and learning actions and activities in mathematics has gained traction in the mathematics education research field. Researchers employ semiotic constructs in furthering the understanding of processes involved in the learning and teaching of mathematics. A semiotic perspective on mathematical activity provides a different lens through which to examine the teaching and learning of geometry in schools. Such a perspective may provide a means by which to make visible the underlying processes and mechanisms of the construction and development of meaning regarding mathematical constructs and processes. As Mudaly (2014:12) notes: "An important aspect to highlight as an overall perspective is the idea that teachers must themselves be cognisant of the semiotics that they engage in together with the language they use."

More importantly, as Iori (2018:112) suggests, it is important that the topic of semiotics is included in pre- and in-service teacher training:

> Hence, there is a need for a professional review of the role semiotic handling plays in the cognitive construction of the mathematical objects and in the assessment of learning processes. Indeed, we believe that this study may open a window not only on the world of research in mathematics education but also on the world of mathematics teacher training, by suggesting a specific professional teacher training on the semio-cognitive processes underlying (the) mathematical activity.

As indicated before, this awareness of semio-cognitive processes would contribute towards a fuller understanding of the complexities involved in the teaching and learning of school geometry. In conjunction with others the semiotic perspectives might open avenues to further exploration of new strategies, techniques and tactics by which to address the current unsatisfactory performance in geometry that manifests also in high-stakes examinations.

Acknowledgement

This research was supported by the National Research Foundation (NRF) in South Africa under grant number 77941. Any opinions, findings and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of the NRF of South Africa.

Authors' Contribution

CRS collected the data and wrote the manuscript. CJ edited the manuscript and assisted with the literature review. FG assisted with the data analysis and the formulation of the conclusion. All authors reviewed the final manuscript.

Notes

i. A pilot version of this article was presented at the Education Association of South Africa (EASA) conference in Durban, January 2018.
ii. Published under a Creative Commons Attribution Licence.
iii. DATES: Received: 9 April 2018; Revised: 16 June 2019; Accepted: 10 August 2019; Published: 31 May 2020.

References

Adeniji SM, Ameen SK, Dambatta BU & Orilonise R 2018. Effect of mastery learning approach on senior school students' academic performance and retention in circle geometry. *International Journal of Instruction*, 11(4):951–962. Available at https://files.eric.ed.gov/fulltext/EJ1191669.pdf. Accessed 3 April 2020.

Atebe HU & Schäfer M 2009. The face of geometry instruction and learning opportunities in selected Nigerian and South African high schools. In M Schäfer & C McNamara (eds). *Proceedings of the seventeenth annual meeting of the Southern African Association for Research in Mathematics, Science and Technology Education: 19 – 22 January 2009*. Grahamstown, South Africa: Rhodes University. Available at https://saarmste.org/images/Conference_Proceedings/SAARMSTE2009-Rhodes_University/Proceedings%20-

%20Book%20of%20Abstracts.pdf#page=34. Accessed 3 April 2020.
Chandler D 2017. *Semiotics: The basics* (3rd ed). New York, NY: Routledge.
Cheah UH, Herbst PG, Ludwig M, Richard PR & Scaglia S 2017. Topic study group no. 13: Teaching and learning of Geometry—secondary Level. In G Kaiser (ed). *Proceedings of the 13th International Congress on Mathematical Education (ICME-13)*. Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-62597-3
Clements DH & Battista MT 1992. Geometry and spatial reasoning. In DA Grouws (ed). *Handbook of research on mathematics teaching and learning*. New York, NY: Macmillan.
Cooney TJ, Davis EJ & Henderson KB 1975. *Dynamics of teaching secondary school mathematics*. Boston, MA: Houghton Mifflin.
Department of Basic Education, Republic of South Africa 2011a. *Curriculum and Assessment Policy Statements Grades 7-9: Mathematics*. Pretoria: Author. Available at https://www.education.gov.za/Portals/0/CD/National%20Curriculum%20Statements%20and%20Vocational/CAPS%20SP%20%20MATHEMATICS%20GR%207-9.pdf?ver=2015-01-27-160141-373. Accessed 15 November 2017.
Department of Basic Education, Republic of South Africa 2011b. *National Curriculum Statement (NCS) Curriculum and Assessment Policy Statement: Further Education and Training Phase Grades 10-12: Mathematics*. Pretoria: Author. Available at http://dsj.co.za/wp-content/uploads/PDF/FET-_-MATHEMATICS-_-GR-10-12-_-Web1133.pdf. Accessed 15 November 2017.
Department of Basic Education 2018. *National Senior Certificate: Diagnostic report*.
Dimmel JK & Herbst PG 2015. The semiotic structure of geometry diagrams: How textbook diagrams convey meaning. *Journal for Research in Mathematics Education*, 46(2):147–195. https://doi.org/10.5951/jresematheduc.46.2.0147
Duval R 2000. Basic issues for research in mathematics education. In T Nakahara & M Koyama (eds). *Proceedings of the 24th conference of the International Group for the Psychology of Mathematics Education (PME 24): Hiroshima Japan, July 23-27, 2000* (Vol. 1). Hiroshima, Japan: Department of Mathematics Education, Hiroshima University. Available at https://files.eric.ed.gov/fulltext/ED452031.pdf. Accessed 3 April 2020.
Duval R 2006. A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61:103–131. https://doi.org/10.1007/s10649-006-0400-z
Duval R 2017. Registers of semiotic representations and analysis of the cognitive functioning of mathematical thinking. In TMM Campos (ed). *Understanding the mathematical way of thinking – The registers of semiotic representations*. Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-56910-9_3
Elia I & Gagatsis A 2003. Young children's understanding of geometric shapes: The role of geometric models. *European Early Childhood Education Research Journal*, 11(2):43–61. https://doi.org/10.1080/13502930385209161
Ernest P 2006. A semiotic perspective on mathematical activity: The case of number. *Educational Studies in Mathematics*, 61:67–101. https://doi.org/10.1007/s10649-006-6423-7
Fey JT (ed.) 1984. *Computing and mathematics: The impact on secondary school curricula*. Reston, VA: National Council of Teachers of Mathematics.
Fischbein E 1993. The theory of figural concepts. *Educational Studies in Mathematics*, 24:139–162. https://doi.org/10.1007/BF01273689
Giannakopoulos A 2017. An alternative way of solving geometry riders in Grade 12: Back to synthesis and analysis. In T Penlington & C Chikiwa (eds). *Proceedings of the 23rd Annual National Congress of the Association for Mathematics Education of South Africa* (Vol. 2). Johannesburg, South Africa: Association for Mathematics Education of South Africa (AMESA). Available at http://www.amesa.org.za/AMESA2017/Volume2.pdf#page=29. Accessed 18 April 2020.
Herbst PG 2006. Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education*, 37(4):313–347. https://doi.org/10.2307/30034853
Horsman R 2019. International perspectives on the teaching and learning of geometry in secondary schools. *Research in Mathematics Education*, 21(1):96–100. https://doi.org/10.1080/14794802.2018.1531055
Iori M 2017. Objects, signs, and representations in the semio-cognitive analysis of the processes involved in teaching and learning mathematics: A Duvalian perspective. *Educational Studies in Mathematics*, 94:275–291. https://doi.org/10.1007/s10649-016-9726-3
Iori M 2018. Teachers' awareness of the semio-cognitive dimension of learning mathematics. *Educational Studies in Mathematics*, 98:95–113. https://doi.org/10.1007/s10649-018-9808-5
Jojo Z 2017. Disrupting a learning environment for promotion of geometry teaching. *Africa Education Review*, 14(3-4):245–262. https://doi.org/10.1080/18146627.2017.1314175
Jones K & Tzekaki M 2016. Research on the teaching and learning of geometry. In A Gutiérrez, GC Leder & P Boero (eds). *The second handbook of research on the psychology of mathematics education: The journey continues*. Rotterdam, The Netherlands: Sense.
Julie C 2016. *Does a CPD initiative focusing on the development of teaching to enhance achievement outcomes in high-stakes mathematics examinations work?* (A report on the first 5-year phase of the Local Evidence-Driven Improvement of Mathematics Teaching and Learning Initiative – LEDIMTALI). Bellville, South Africa: University of the Western Cape.
Kuzniak A 2018. Thinking about the teaching of geometry through the lens of the theory of geometric working spaces. In P Herbst, U Cheah, P Richard & K Jones (eds). *International perspectives on the teaching and learning of*

geometry in secondary schools. Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-77476-3_2

Marchis I 2012. Preservice primary school teachers' elementary geometry knowledge. *Acta Didactica Napocensia*, 5(2):33–40. Available at https://files.eric.ed.gov/fulltext/EJ1054293.pdf. Accessed 21 March 2020.

Marczyk G, De Matteo D & Festinger D 2005. *Essentials of research design and methodology*. Hoboken, NJ: John Wiley & Sons.

Martínez JM & Dominguez H 2018. Navigating mathematics and language tensions in language immersion classrooms. *Teaching and Teacher Education*, 75:1–9. https://doi.org/10.1016/j.tate.2018.05.013

Merrel F n.d. *Semiotics versus semiology: Or, how can we get a handle on semiosis?* Available at http://www.digitalpeirce.fee.unicamp.br/floyd/p-semflo.htm. Accessed 15 November 2017.

Moro L, Mortimer EF & Tiberghien A 2019. The use of social semiotics multimodality and joint action theory to describe teaching practices: Two cases studies with experienced teachers. *Classroom Discourse*. https://doi.org/10.1080/19463014.2019.1570528

Morris R (ed.) 1986. *Studies in mathematics education: Geometry in schools* (Vol. 5). Paris, France: UNESCO. Available at https://unesdoc.unesco.org/ark:/48223/pf0000124809. Accessed 8 April 2020.

Mudaly V 2012. Diagrams in mathematics: To draw or not to draw? *Perspectives in Education*, 30(2):22–31.

Mudaly V 2014. A visualisation-based semiotic analysis of learners' conceptual understanding of graphical functional relationships. *African Journal of Research in Mathematics, Science and Technology Education*, 18(1):3–13. https://doi.org/10.1080/10288457.2014.889789

O'Halloran KL 2011. The semantic hyperspace: Accumulating mathematical knowledge across semiotic resources and modalities. In F Christie & K Maton (eds). *Disciplinarity: Functional linguistic and sociological perspectives*. New York, NY: Continuum International Publishing Group.

Otte M 2006. Mathematical epistemology from a Peircean semiotic point of view. *Educational Studies in Mathematics*, 61:11–38. https://doi.org/10.1007/s10649-006-0082-6

Presmeg N, Radford L, Roth WM & Kadunz G 2018a. Discussion and conclusions. In N Presmeg, L Radford, WM Roth & G Kadunz (eds). *Signs of signification: Semiotics in mathematics education research*. Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-70287-2

Presmeg, N, Radford L, Roth WM & Kadunz G (eds.) 2018b. *Signs of signification: Semiotics in mathematics education research*. Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-70287-2

Radford L, Demers S, Guzmán J & Cerulli M 2003. Calculators, graphs, gestures and the production of meaning. In P Pateman B Dougherty & J Zilliox (eds). *Proceedings of the 27 conference of the international group for the psychology of mathematics education (PME27 –PMENA25)* (Vol 4). Honolulu, Hawaii: University of Hawaii. Available at https://telearn.archives-ouvertes.fr/hal-00190515/document. Accessed 18 April 2020.

Renne CG 2004. Is a rectangle a square? Developing mathematical vocabulary and conceptual understanding. *Teaching Children Mathematics*, 10(5):258–263.

Sáenz-Ludlow A & Kadunz G (eds.) 2016. *Semiotics as a tool for learning mathematics: How to describe the construction, visualization, and communication of mathematical concepts*. Rotterdam, The Netherlands: Sense. https://doi.org/10.1007/978-94-6300-337-7

Sáenz-Ludlow A & Presmeg N 2006. Guest editorial: Semiotic perspectives on learning mathematics and communicating mathematically. *Educational Studies in Mathematics*, 61(1/2):1–10. https://doi.org/10.1007/s10649-005-9001-5

Sinclair N & Moss J 2012. The more it changes, the more it becomes the same: The development of the routine of shape identification in dynamic geometry environment. *International Journal of Educational Research*, 51–52:28–44. https://doi.org/10.1016/j.ijer.2011.12.009

Suydam MN 1985. The shape of instruction in geometry: Some highlights from research. *The Mathematics Teacher*, 78(6):481–486.

Taylor SJ, Bogdan R & DeVault ML 2016. *Introduction to qualitative research methods: A guidebook and resource*. Hoboken, NJ: John Wiley & Sons.

Zakariyya AA, Ndagara AS & Yahaya AA 2016. Effects of mastery learning approach on senior secondary school students' performance in geometry in Bida, Niger state, Nigeria. *ABACUS: Journal of the Mathematical Association of Nigeria (MAN)*, 41(1):17–25.