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Affordances for developing mathematical proficiency in fraction word problems in South African textbooks

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Background: Performance in fraction word problems in primary school indicates that learners face challenges when working with fraction word problems, despite their frequent use of textbooks as learning materials.

Aim: This article presents a comparative analysis of three commonly used Grade 6 mathematics textbooks, with the objective of evaluating their potential to facilitate the development and promotion of mathematical proficiency in fraction word problems among learners.

Methods: Fraction word problems examples are analysed and compared across three commonly used Grade 6 textbooks using the five strands of mathematical proficiency.

Results: Findings across the textbooks revealed an emphasis on the promotion of procedural fluency and productive disposition. The under-representation of the other strands, however, may hinder a holistic development of mathematical proficiency in fraction word problems and hence limit learners' exposure to a diverse range of word problems in the textbooks.

Conclusion: While the three selected Grade 6 mathematics textbooks are intended to promote the aims of the curriculum, their presentation on fraction worked-out examples and fraction word problems compromises an all-encompassing development of proficiency. This finding is pertinent to textbook developers as South Africa edges towards a new curriculum that will entail new textbook development.

Contribution: The study's results address a significant practical gap in the three selected Grade 6 textbooks regarding the development of mathematical proficiency in the presentation of fraction word problems. The study also makes a methodological contribution by operationalising the five strands of mathematical proficiency for use in the analysis of examples (in textbooks).

Keywords: fractions; fraction word problems; mathematics textbooks; mathematical proficiency; examples in mathematics.

Introduction

According to the 2014 Annual National Assessment (ANA) results, the general mathematics performance of South African learners was documented to have been below 60%, and specifically, Gauteng's Grade 6 mathematics performance was documented to be at 29% (Department of Basic Education [DBE] 2014). A notable discrepancy emerged between Grade 6 and Grade 9 ANA results: Grade 6 learners demonstrated some proficiency in fraction concepts, contrasting greatly with the Grade 9 results, which identified fractions as a weakness. This disparity raises a critical question of whether learners have mastered their fraction concepts and word problem-solving in Grade 6. In general, research has documented a persistent challenge in learners' understanding of fractions in primary school mathematics (Doğan & Tertemiz 2020; Ubah 2021). Learners' difficulties with the comprehension of fractions are substantially attributed to the abstract nature of fractions (Wilensky 1991). Moreover, the challenges learners encounter with fraction comprehension in primary school extend into high school mathematics, constituting a cumulative obstacle to progress where the mastery of fractions is essential for mathematical progression (Bruce et al. 2013). Stewart (2005) posits that the curriculum's initial exposure of fractions to learners is often limited to halves and quarters at the expense of other fractional representations. This limited exposure diminishes the opportunity for learners to make connections of the fraction concepts in their lives. As a result, this exacerbates the abstractness of learning fractions with comprehension.

In addition to the challenges learners face with fraction comprehension, solving fraction word problems has been observed to pose an equal challenge to learners (Mokhtar et al. 2019). Fraction word problems are mathematical representations of real-world situations that involve the use of fractions, but as Alfonso, García and Gabarda (2016) notice, fraction word problems may also use pseudo-realistic narratives that are not necessarily applicable to a particular real-world situation. Contrary to the intended motivational role of word problems in encouraging mathematical engagement and enthusiasm for learners (Brehmer, Ryve & Van Steenbrugge 2015), our observation is that Grade 6 learners exhibit persistent challenges with word problems, especially those involving fractions. Stewart (2005) argues that learners struggle with fractions because they lack the ability to connect fraction word problems to their lived experiences. Such a deficit is concerning given: (1) the South African curriculum's emphasis on common fractions, which is covered across two terms per grade in the intermediate phase (Grades 4–6); (2) the inclusion of dedicated sections on fraction word problems in textbooks and (3) the mandatory inclusion of fraction word problems in assessments. With regard to the inclusion of sections on fraction word problems in textbooks (on which this article focuses), research has long acknowledged the importance of examples (that is, the mathematics questions and/or tasks that illustrate a particular concept in mathematics) in enabling or constraining access to mathematical knowledge (Bills et al. 2006; Essien 2021; Renkl 2017). These research studies are categorical in showing that the type of examples that teachers choose, or the type of examples that are included in (or excluded from) textbooks, can offer different learning opportunities or affordances to learners. This is so because learners depend on textbooks as resources to mitigate their struggles when working with mathematical concepts, particularly when they do not have sufficient proficiency. In so doing, learners find themselves relying on worked-out example procedures to make sense of how they should go about solving fraction word problem exercise examples presented to them.

According to Kilpatrick, Swafford and Findell (2001), textbooks should thoroughly support the comprehensive development of all five strands of mathematical proficiency. This implies that the development of fractions essentially involves learners' development of their conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Therefore, the purpose of this article is to investigate the affordances of fraction worked-out examples, and fraction word problem exercise examples in three commonly used Grade 6 Curriculum Assessment and Policy Statement (CAPS) approved textbooks: *Headstart Mathematics Grade 6* (Facer, Kruger & Pretorius 2012), *Platinum Mathematics Grade 6* (Bowie et al., 2012) and *Shuters Premier Mathematics Grade 6* (Tiaden, Farrell & John 2013).

This study seeks to address the following questions:

- *How do three commonly used Grade 6 mathematics textbooks (Headstart, Premier Mathematics and Platinum Mathematics) enable the development of the five strands of mathematical proficiency in fraction word problems*?
- *What affordances for learning about fraction word problems are offered in these Grade 6 textbooks*?

To address these research questions, this article utilises Kilpatrick et al.'s (2001) strands of mathematical proficiency as a framework to analyse how mathematical proficiency is promoted in fraction word problems, worked-out examples and word problems exercise examples. Notably, these strands also informed the Teaching Mathematics with Understanding (TMU) Framework of the DBE (2018). This article contributes to knowledge in two ways: Firstly, while there has been substantial research into fractions, word problems and fraction word problems, there is a dearth of research into the learning opportunities that textbooks in themselves make available to learners for understanding fraction word problems. This study addresses this gap. Secondly, the seminal work of Kilpatrick et al. (2001) has been used extensively in research, as with this study. What this study does differently is to provide a methodological approach (using Kilpatrick et al.'s [2001] strands of mathematical proficiency) for analysing mathematics examples, thus providing a new lens through which textbook examples can be evaluated. In so doing, this study makes a methodological contribution to the existing framework on mathematical proficiency with respect to the analysis of examples in mathematics.

Fraction word problems in mathematics

The teaching and learning of fractions as a mathematical concept encompasses a large variety of interrelated mathematical concepts and knowledge that extend to the next grades (Bruin-Muurling 2010). An extensive amount of research has documented that learners continuously demonstrate a limited understanding of fraction concepts (Behr et al. 1983; Lortie-Forgues, Tian & Siegler 2015; Stigler, Givvin & Thompson 2010) despite intense instructions (Lortie-Forgues et al. 2015) and multiple interventions (Greer 2008; Lovemore, Robertson & Graven 2021). One of the key reasons why both teachers and learners find fractions difficult is what Ni and Zhou (2005) termed whole number bias – when learners work with fractions the same way they would when they are working with properties of whole numbers. This bias poses an interference between the learners' prior knowledge and the construction of new knowledge with regard to learning the concept of fractions. Furthermore, Gabriel et al. (2013) have documented that there are few core differences between fractions and whole numbers, and when these are not addressed thoroughly, they have the potential to lead to whole number bias and make the overall learning experience of fractions challenging. Neagoy (2017) argues that the introduction of fractions requires a crucial cognitive shift in how learners think about fractions. To address this transition, Neagoy (2017) proposes two key pedagogical approaches: the first one is teacher awareness and sensitivity concerning the transition from additive to multiplicative thinking, and the second transition is from whole numbers to rational numbers. Learners are used to thinking in the

additive path because of how their early experiences of numbers are rooted in adding and subtracting. However, for learners to successfully engage with per cent, ratio and proportion:

> '[*M*]ultiplicative thinking is a prerequisite for working with these powerful and necessary ideas. Students cannot be expected to understand and use rational number ideas and representations with any confidence if their understanding of multiplication (and division) is restricted to a 'groups of' model with small whole numbers.' (Siemon, Breed & Virgona 2005:1)

Consequently, learners need to demonstrate the cognitive capacity to recognise multiplicative and division problem structures in word problems and apply appropriate algorithms and strategies.

According to Siegler, Thompson and Schneider (2011), to become proficient with fractions, learners are required to develop sound conceptual and procedural knowledge, which encompasses the ability to apply procedures efficiently. In addition to the two types of knowledge that are expected to take place in the development of fraction proficiency, Siegler et al. (2011) suggest that the development of symbolic and non-symbolic knowledge is just as important, as it enables learners to examine fractional concepts using different representations. In a study conducted by Peck and Jencks (1981) with Grade 6 learners, findings indicated that learners who had gaps in their fraction knowledge often struggled with applying previously learned rules in varying contexts correctly. The study revealed that learners who struggled with the application of rules had knowledge gaps that affected their reasoning, resulting in their dependency on their teachers' validation to determine the accuracy of their answers.

Word problems in mathematics present an even greater cognitive challenge for bilingual learners because of language complexities (Ahmad & Jusoff 2009; Madzorera & Essien 2018; Tong & Loc 2017). Successful word problem-solving requires learners to have the ability to integrate conceptual understanding and procedural fluency to problem solve (Madzorera & Essien 2018). Therefore, mathematical proficiency is a prerequisite for effective engagement with word problems.

Examples in mathematics textbooks

Textbooks are considered a very resourceful element in the teaching and learning of mathematics and a very common feature in mathematics classrooms (Okeeffe 2013). Literature has indicated that learners use mathematics textbooks independently without any guidance from their teacher, making them an equally influential and important element of mediation (Kristanto & Santoso 2020; Rezat, Fan & Pepin 2021). Mathematics textbooks consist of various elements that aid towards the learning of mathematics, such as explanatory texts, worked-out examples and exercise examples. Examples (both worked-out and exercise) are considered as an important element in the teaching and learning of mathematics, and as a result, research (Bills et al. 2006) has suggested that special attention needs to be given to understanding their usefulness.

Worked-out examples are mathematics questions or tasks that textbooks use to demonstrate step-by-step procedures of how to solve the problem from an 'expert's model' (Atkinson et al. 2000). Effective worked-out examples are expected to contain and demonstrate sufficient information to allow learners to learn through them without the need for additional instructional information (Zhu & Simon 1987). On the other hand, exercise examples are mathematics questions/tasks that a textbook provides for learners to solve independently. Together, both worked-out examples and exercise examples make up a significant part of mathematics textbooks, and as such, need to be key tools for scaffolding and supporting learners' development of mathematical proficiency. With this in mind, the analysis of examples (both worked-out and exercise examples) in textbooks is crucial as it offers insights into the learning opportunities or affordances for the development of mathematical proficiency that are made available to learners who use these textbooks.

Kilpatrick et al.'s (2001) five strands of mathematical proficiency

This study is conceptually and analytically framed by the five strands of mathematical proficiency, and it has been adapted for the analysis of the textbooks used for this study. The five strands of mathematical proficiency were originally developed with extensive research in cognitive psychology, mathematics education and research to provide an overview of a successful, composite and comprehensive learning process of mathematics (Kilpatrick et al. 2001). Kilpatrick et al.'s view of what successful learning of mathematics encompasses is explained by their decision to term it mathematical proficiency. Kilpatrick et al.'s (2001) notion of the five strands of mathematical proficiency is relevant to this study because the framework emphasises the competence, in our case, fraction competence, which is necessary to successfully learn mathematics. As Kilpatrick et al. (2001) argue, for mathematical proficiency to be an attainable educational goal, instructional programmes need to advocate and address the development of all the strands. By extension, we contend that a textbook's objective is to promote competence in the understanding of mathematical concepts, and to do this, textbooks need to demonstrate the development and promotion of all five strands, not just a select few, through the use of the textbooks. The five strands of mathematical proficiency are elaborated as follows:

Conceptual understanding is the comprehension of mathematical concepts and how they are connected and related to one another (Kilpatrick et al. 2001). The possession of conceptual understanding is demonstrated by having knowledge that is beyond knowing isolated facts but extends to its applications and its mathematical use. We will engage further with conceptual understanding as it concerns our study in a subsequent section.

Procedural fluency is the knowledge of executing procedures accurately, appropriately and efficiently (Kilpatrick et al. 2001). Kilpatrick et al. argue that procedural fluency 'also

supports the analysis of similarities and differences between methods of calculating' (Kilpatrick et al. 2001:121). Comprehensive understanding of concepts is considered the basis for procedural fluency and is therefore regarded as the skill gained from understanding a concept in depth.

Strategic competence is defined as 'the ability to formulate mathematical problems, represent them, and solve them' (Kilpatrick et al. 2001:124). The strategic competence strand caters for the fact that learners encounter problems outside the classroom that will require mathematics to be used. In schools, strategic competence is developed through problem-solving, especially in mathematical problems that are non-routine in nature.

Adaptive reasoning is defined as the ability and skill to logically provide clarification and justification. This strand develops when learners have enough knowledge about the task, have a good understanding of what is required from them and are able to explain and/or justify their thinking.

Productive disposition is the ability to see mathematics as being sensible, useful, and one's self-belief in their potential. This strand develops from giving learners opportunities to experience mathematics and apply their knowledge in real-life contexts, giving them the opportunity to see and make their own sensible meaning of mathematics and its usefulness.

Kilpatrick et al. (2001) argue that these five strands are interwoven and inextricably intertwined to achieve full mathematical proficiency.

The study

This research study uses a qualitative content analysis research approach. The qualitative aspect of this research aims to provide an in-depth understanding of the textbooks' affordances offered to learners by studying the evidence that comes from the results yielded from the examination of the three Grade 6 sampled textbooks. Kilpatrick et al.'s (2001) idea of mathematical proficiency is used in this study to tease out the extent to which textbooks promote fraction proficiency. The unit of analysis for this study is fraction word problem examples from three selected Grade 6 textbooks.

Selection of textbooks and the choice of fractions as a topic

As indicated previously, the three selected textbooks for this analytical study are Headstart (Facer et al. 2012), Platinum (Bowie et al. 2012) and Shuters Premier (Tiaden et al. 2013). These textbooks were selected because they are CAPS-approved, commonly used textbooks in Grade 6 and a major mathematics teaching resource. The CAPS aims to ensure that learners in Grade 6 are competent to 'solve problems in contexts involving fractions, including grouping and sharing' (DBE 2011:16). The contents on fractions in all three analysed textbooks are aligned with the curriculum's goal of ensuring that learners are proficient in performing calculations with fractions.

How the five strands of mathematical proficiency were adapted for textbook analysis

The five strands of mathematical proficiency were not originally designed for the analysis of textbook questions (examples), but as textbooks are instructional resources, our contention is that they have the potential to promote the five strands of mathematical proficiency. What this meant for us was that some work was needed in developing a methodological approach (based on the Strands of mathematical proficiency) that can be used for analysing examples, and in particular for analysing fraction word problem examples in textbooks.

The emergence of a methodological approach based on the strands of mathematical proficiency

We first needed to make an adaptation to Kilpatrick et al.'s framework by extending the definitions of the five strands of mathematical proficiency to make them applicable to the analysis of fraction word problem examples. For instance, Kilpatrick et al. (2001) define strategic competence as 'the ability to formulate mathematical problems, represent them, and solve them' (p. 118). However, for this study, this definition is adapted to talk to and investigate how strategic competence is promoted in the textbooks. As a result, strategic competence in textbooks is defined as the promotion of the formulation, representation and solution of mathematical problems. The essence of the original definition is still maintained in the adapted definition. Subsequently, all the definitions of the five strands are adapted to suit their new task of the framework that is to investigate the promotion of mathematical proficiency.

Given the nature of the original framework (Kilpatrick et al. 2001), it is anticipated that certain fraction word problem exercises and worked-out examples will inherently promote multiple strands of mathematical proficiency. For instance, a single question might simultaneously promote procedural fluency and strategic competence, among other strands. To account for this complexity, our analysis meticulously recorded all the strands that each fraction word problem exercise or worked-out example promotes, rather than attempting to force them to be categorised under a specific single strand. This approach acknowledges that the strands work together and support each other (Kilpatrick et al. 2001) for the development of a comprehensive mathematical understanding. In addition, the approach acknowledges the multifaceted nature of mathematical word problems, which often requires an interplay of different mathematical competences. If a fraction word problem exercise or worked-out example fits into multiple strands, it was classified as such, without any attempt to constrain it to a single category. By capturing all relevant strands, we aim to develop a nuanced understanding of the affordances from each

textbook, reflecting the interconnected and holistic nature of mathematical proficiency. In what follows, we present each strand and the indicators we used to analyse them.

Procedural fluency

For procedural fluency, our methodological approach explores the extent to which there is potential promotion of this strand in the textbooks. Consequently, procedural fluency is subdivided into three categories: the first category is termed the *1st degree of procedural fluency* or PF^{+} (these will be used interchangeably throughout this article). The 1st degree of procedural fluency is identified by the use of only general vocabulary in everyday English. These are signifiers that are used daily and may be common in learners' everyday speech. Examples in this category offer a suggestive pathway to solving a word problem because the language in the examples is used in everyday speech, making it easy to mathematically translate them to the correct operation when problem-solving. A further elaboration of what will be regarded as verbal cues (or cue words) for this analytical study is informed by Jerman and Rees' (1972) conception of verbal cues – that is, words that are a cue or rather a guide for an operation to be used. Jerman and Rees' (1972) notion of verbal cues refers to words such as *and*, *left*, *each*, *average* and altogether as verbal cues for addition, subtraction, multiplication and division. However, Jerman and Rees (1972) recognise that 'verbal cues' can be distracting. 'Distractors' are verbal cues that are not cues for the intended operation that should be used. A very common distractor is 'each', as it can suggest a multiplication or division pathway. Such distractors do still suggest a pathway for learners to follow. However, they (learners) must use their mathematical thinking to apply the correct operation in problem-solving.

Therefore, only everyday words that align with the mathematical language are considered as promoting procedural fluency in the first degree. Cues in the everyday English language are also regarded as verbal cues. It is important to mention that we have also taken into consideration how everyday English language verbal cues may suggest incorrect computation pathways depending on context and learners' interpretation of the word problem. For this study, we used the notion of everyday English and mathematical language verbal cues used in the study by Jerman and Rees (1972) to predict the relative difficulty of verbal arithmetic problems.

Also, everyday English vocabulary is considered to offer a suggestive pathway when it is used and interpreted in the right context. For example, the verb 'eat' used in a word problem could possess various meanings, but once its context is taken into consideration, the suggestive pathway will become apparent. For example, in the question: 'Two friends share a chocolate bar divided into 12 pieces, the first friend eats $\frac{2}{3}$ and the second friend eats $\frac{1}{6}$ of the chocolate', if the first question asks 'how many pieces did the first friend eat?', this is a question of finding a fraction of a whole. However, if the second question asks learners to calculate how much chocolate the two friends eat altogether, learners will have to find the sum of the two fractions to solve the problem. Another question would be who ate the most chocolate? And this will be a comparison of the two fractions representing the pieces eaten. The point here is that no assumption about a verbal cue should be made, and all general everyday language cues are analysed according to their intended context.

Next is the *2nd degree procedural fluency (or PF^{++}).* Unlike the previous degree of procedural fluency, this degree is identified by a different kind and a more mathematical form of vocabulary. This kind of vocabulary is regarded as mathematical terms – a word that has a specific mathematical meaning in the context it is used in, which can be both technical or special. The difference between technical and special mathematical terms is that technical terms are specialised terms in the field of mathematics. On the other hand, special words are used on a daily basis but can convey a different special or mathematical meaning in the context of mathematics, e.g. 'match', 'set', 'group', 'figure'. It is important to bring up the distinction between the 1st degree and 2nd degree of procedural fluency beyond the presented verbal cues. Flexibility, efficiency and accuracy in procedures become more developed with the presence of mathematical terminology in addition to everyday English verbal cues. In other words, with the conceptual understanding of mathematical vocabulary that is routinely used in mathematics textbooks and word problems, learners get access to mathematical tools that enable them to learn mathematics (Miller 1993).

Unlike the first two degrees of procedural fluency, the *3rd degree of procedural fluency (PF^{+++})* is a routine problem that does not contain any obvious form of verbal cues to the solution of the problem.

The last degree in the procedural fluency strand is the *4th degree of procedural fluency (PF^{++++}).* As we will elaborate further under strategic competence, a fraction word problem example that is non-routine and has no straightforward pathway to solving it is categorised as promoting strategic competence. However, if the same kind of fraction word problem example appears for the second time, it is regarded as promoting procedural fluency because the second exposure to the fraction word problem example offers learners the skill sets and cognitive tools to solve the problems more flexibly and more accurately. The fraction word problem examples that fit this category are recorded as promoting procedural fluency at the fourth degree, and this is what differentiates this level of procedural proficiency from the first three, which are actually routine problems. It is important to notice that the re-categorisation of fraction word problems from strategic competence to 4th degree of procedural fluency does not imply that the problems are easy to solve.

Strategic competence

All fraction word problem examples that are non-routine and are appearing for the first time in the textbooks are categorised

as promoting strategic competence (*SC*) because they do not offer any suggestive pathway to solving the problem. The adapted version of this strand does not have any degrees because non-routine problems inherently promote strategic competence.

Adaptive reasoning

Fraction word problem examples that probe for further explanation, justification, reasoning, proving or what Essien and Adler (2016) refer to as authorising practices are analysed as promoting adaptive reasoning (*AR*).

Productive disposition

The productive disposition strand is divided into three subcategories or degrees, the 1st being PD^{+} fraction word problem examples, which is when the information or data in the word problem exists in the real world. The problem might involve quantities, measurements, or facts that are true in a real-world context. However, the event or situation described in the problem does not necessarily have to be something that has actually happened or is likely to happen. It could be a hypothetical or even somewhat unrealistic scenario, as long as the numbers or data themselves are grounded in reality.

The 2nd degree, PD^{++}, is identified by the evidence of real data being used and the event described having taken place or having the potential to take place in the real world. This means the scenario is not just using real numbers, but it also depicts a realistic occurrence or possibility. For example, the recipe uses real ingredients with real measurements (e.g., $\frac{1}{2}$ cup of flour, 2 eggs), and if the recipe calls for baking at a temperature of 180 degrees Celsius for 15 min. Both the data exist in the real world and the scenario depicts a realistic occurrence. The *data* (ingredients and measurements) are real and the *event* (baking process or result) has a chance of occurring in the real world. The last degree is PD^{+++}, where the data and the scenario created are exaggerated for the learning of mathematical concepts and the development of number sense. According to Mohamed and Johnny (2010), learners who have mastered their number sense are able to perform mental computations with flexibility and 'appropriateness sense of reasonableness' (p. 1). PD^{+++} impacts mathematical fluency as the exaggerated data and scenario develops the intuition to recognise the direct link between a quantity and the number that represents it. This aspect of proficiency is fundamental to higher-level mathematical thinking that is required in understanding population growth, disease spread modelling, financial data and energy production/consumption. Therefore, to confidently and successfully work with exaggerated numbers provides learners with tangible evidence of their mathematical abilities.

It is critical to observe that the progression of the subdivisions in procedural fluency and productive disposition does not necessarily imply an increase in difficulty. For example, the same word problem can be presented in ways that either contain general verbal cues or mathematical verbal cues or both. The argument is that the varying use of verbal cues has the potential to promote procedural fluency to varying degrees. Procedural fluency is not solely tied to accurately performing calculations but to knowing when and why they are used (Kilpatrick et al. 2001). Therefore, the progression of the subdivisions in procedural fluency leads to a shift from assisting learners to connect mathematical concepts to real-life situations to helping them understand and apply mathematical terminology in varying scenarios. In other words, an extensive exploration of procedural fluency at varying degrees has the potential to promote flexibility, accuracy and efficiency in the development of knowledge procedures.

To account for the different strands of mathematical proficiency and their degrees, a two-level iterative coding process was employed to organise the worked-out and exercise examples across the three textbooks. This process involved shifting between inductive and deductive approaches. Initially, the coding categories were deductively drawn from the strands of mathematical proficiency. We then selected a total of 30 questions – 10 from each textbook – for individual analysis and preliminary coding. After this, we convened to compare and discuss our individual codes, which led to the refinement of the category descriptors. These refined descriptors were then used by an independent researcher to code the same set of 30 questions. Following this, we met again to collaboratively review and discuss the outcomes, which informed the coding of the remaining questions. Throughout this ongoing process, we further refined the descriptors, culminating in the final version presented in Table 1. It must be noticed that the examples used to illustrate the categories of the strands are not from the textbooks. We developed examples that are representative of those found in the analysed textbooks, demonstrating the framework in action.

The issue of conceptual understanding

Like Askew (2012), our contention is that conceptual understanding can be regarded as a primary strand for the other strands to come into manifestation. Askew (2012) argues that proficiency in mathematics is an action students need to engage in and be experts in when learning and using content. This implies that the strands of mathematical proficiency are action words – verbs. Hence, according to Askew (2012), conceptual understanding cannot be regarded as something learners 'do' towards developing mathematical proficiency. It has to be taken as the results of the other strands. In other words, conceptual understanding cannot be observed as it plays out. For our methodological approach and the reasons aforementioned, conceptual understanding is assumed to be promoted in all word problem examples, and as such is not represented as a category in Table 1.

TABLE 1: Definitions and recognition rules for the strands of mathematical proficiency.

Strands	Definition	Description	Categories of description	Rules	Categorisation Justification
Procedural fluency	The promotion of flexibility, accuracy and efficiency in the development of knowledge procedures	This is seen when a word problem is a routine problem.	*PF*$^{+}$: contains general verbal cues Example: Molly saved $\frac{1}{4}$ of the money that she earned and spends the rest. If she earned R100, how much did she spend? *PF*$^{++}$: contains mathematical verbal cues or both mathematical and general verbal cues Example: At the end of the week, a farmer expects to harvest $\frac{7}{10}$ of his bean crop altogether. If he has already harvested $\frac{1}{4}$ of the crops, what difference in the crop remains to be harvested this week? *PF*$^{+++}$: contains obvious suggestive pathways Example 1: In a bouquet of 24 red and white flowers, $\frac{1}{3}$ of the flowers are red. How many of the flowers in the bouquet are white? *PF*$^{++++}$: If a strategic competence question appears for the second time, it becomes a routine problem	*PF*$^{+}$: everyday verbal cues have to be words used in the everyday English language. They should provide a clear and concise pathway to solving the problem in the correct context. *PF*$^{++}$: mathematical verbal cues have to be mathematical cues that are commonly used in mathematics as described by Jerman and Rees (1972: 309) 'This conception of verbal cues included words such as "added", "altogether", "gained", "how much less", "lost", "left", "each", "average"' *PF*$^{+++}$: suggestive pathways are unclear as they cannot be classified as everyday English or mathematical verbal cues, but the problem is a routine problem.	'By studying algorithms as "general procedures," students can gain insight into the fact that mathematics is well structured (highly organised, filled with patterns, predictable) and that a carefully developed procedure can be a powerful tool for completing routine tasks' Kilpatrick et al. (2001). -Saved, earned, spent and rest are everyday English cues. • Added,altogether, gained and divided are clear mathematical verbal cues that do not possess any ambiguity in the mathematical context. • Finding a fraction of a whole is a routine exercise, but there is no cue that suggests the exact pathway to be followed.
Strategic competence	The formulation, representation and solution of mathematical problems	This is seen when a word problem is a non-routine question with no straightforward pathway.	*SC*: First time appearance in the textbook. Example: Tom, Sarah and Jack are related. Jack is $\frac{1}{3}$ of Sarah's age and Tom is $\frac{1}{4}$ of Sarah's age. If Tom is 5 years younger than Jack, how old is Sarah?	*SC*: problem is non-routine with no straightforward pathway to solving it. These are Higher Order Problems. The context of the question is set in an abstract world of mathematics and is not limited to real-life contexts.	Robertson argues that if a problem has been solved before using previously taught methods, then that task can be considered an 'exercise' because of its reappearance (Robertson, 2017). Hence, the second time the question is asked, it will be considered as a procedural fluency question.
Adaptive reasoning	The development of the skill of reasoning with logic, explanation and justification.	This is seen when a word problem requires an explanation or justification.	*AR*: contains the requirement to explain, provide proof or justification. Example: Nkanyezi and Alex are planning a trip from Rustenburg to Johannesburg. Their petrol tank is $\frac{4}{7}$ full, and they know they need $\frac{3}{4}$ of their tank to complete the trip. Do they have enough petrol to complete the trip? Show your calculations and justify your answer.	*AR*: The problem's objective is to develop adaptive reasoning through 'explaining, proving or justifying', but this is allowed to be posed in different and creative ways in the problem.	One manifestation of adaptive reasoning is the capacity to provide justification for one's solutions or thinking Kilpatrick et al. (2001).
Productive disposition	The promotion of sensibility, usefulness, worthiness and self-efficacy in mathematics.	This is seen when the everyday life situation embedded in the word problem emulates reality.	*PD*$^{+}$: Information/ data in the word problem exists in the real world. Example: Thandi baked a dozen muffins. She gave $\frac{1}{3}$ of them to her neighbour. How many muffins did Thandi give to her neighbour? *PD*$^{++}$: Real data are used, and the event described has taken place or has the potential to take place. Example: A typical school day in Roodeplaat has 7 h of instruction. Because of a special school event, classes were shortened, and students only had $\frac{2}{3}$ of a typical school day. How many hours of instruction did the students have on that day? *PD*$^{+++}$: Data and the scenario created are exaggerated for the learning of mathematical concepts. Example: On the candy planet of Sweetopia, the Great Gumball Tree produces 1 000 000 gumballs every hour. If a swarm of Sugar Snails manages to slurp up $\frac{15}{10\,000}$ of the hourly gumball production, how many gumballs do these speedy snails devour each hour?	*PD*: Real-world word problems and their information should be problems that have been experienced by learners or have the potential to be experienced. This promotes a higher sense of useful mathematics.	'Realistic' word problems have been documented to have the potential to promote learners' willingness to do mathematics.

PF, procedural fluency; SC, strategic competence; AR, adaptive reasoning; PD, productive disposition.
+, 1st degree; ++, 2nd degree; +++, 3rd degree; ++++, 4th degree.

Ethical considerations

Ethical clearance waiver for the study was granted by the University of the Witwatersrand with the protocol number: 2022ECE014M.

Analysis and discussion of findings

The analysis for each textbook begins with an overview of the number of worked-out examples and word problem exercise examples in each textbook. The examples are then categorised under each strand of mathematical proficiency (as adapted from Kilpatrick et al. 2001) with percentages for comparative purposes in Table 2.

Procedural fluency

Looking at Table 2, procedural fluency in the Premier mathematics textbook is promoted at 54%, 47% in the Platinum and 49% in the Headstart. What insight do these findings from the analysis reveal? These findings suggest that learners using these textbooks are provided with ample opportunities to explore, advance and develop their procedural fluency through engaging with routine fraction word problem exercises. The recurring appearance and engagement of procedural fluency questions in textbooks afford learners the opportunity to practice and refine their thinking, and by so doing, there is an opportunity to focus on areas of weakness that will result in their (learners') development (Foster 2013). Conceptual understanding is a fundamental strand for mathematical proficiency, and procedural fluency builds a foundation that later enables slightly advanced problem solving (Kilpatrick et al. 2001; Mellor, Clark & Essien 2018). In what follows, we disaggregate the promotion of these findings on procedural fluency into their levels as discussed in Table 1.

PF^{+}: To reiterate on the categorisation of examples in this category, all the fraction word problem examples were categorised for their incorporation of everyday cues that offer learners suggestive pathways to the appropriate procedures to employ. In the Premier textbook, the first degree of procedural fluency (*PF*$^{+}$) is accounted for by only six fraction word problem examples (13%). This suggests that only six exercise examples provide an in-depth practice of foundational concepts through routine exercises that promote rules, algorithms and the selection of appropriate procedures to solve fraction word problem exercises. In Platinum, 11% of the fraction word problem exercise examples promote procedural fluency to the first degree (*PF*$^{+}$). Lastly, Headstart has seven fraction word problem exercise examples (13%) that promoted procedural fluency on the first degree (*PF*$^{+}$). Newton, Lange and Booth (2020) argue that the acquisition of fully developed flexibility is rare among primary school learners compared to some progressive high school learners. All three textbooks provide learners with comparable opportunities to encourage and develop flexibility and accuracy through everyday language within the exercise problems.

PF^{++}: Table 2 also reveals that 15% (7 exercise examples) of the fraction word problem exercise examples in the Premier textbook promote procedural fluency to the 2nd degree with the addition of technical mathematical verbal cues. Fraction word problems in this category put an emphasis on the shift from everyday language application to a more technical language. The Platinum textbook offers more affordances to learners than the Premier textbook, with 32% exercise examples that promote procedural fluency at the 2nd degree (*PF*$^{++}$). Learners using the Platinum textbook are exposed to an important aspect of language transitions. The language transition from everyday English to a mathematical language is essential in the learning of fractions or any other mathematical concept, as it aids in familiarity with mathematical terminology and develops precise problem-solving techniques. Headstart also provides a significant amount of *PF*$^{++}$ coverage, with 16 (29%) exercise examples in this category. The opportunities offered in the transition from *PF*$^{+}$ and *PF*$^{++}$ expose learners to a language that is more discipline-specific. Therefore, Platinum and Headstart afford learners more opportunities to learn and use mathematics terminology precisely. This transition is not only critical for the development of mathematics terminology, but as Ilany and Margolin (2010) assert, transition is also important

TABLE 2: Comparative analysis of mathematical proficiency in each textbook.

Mathematical proficiency strands	Codes	Premier mathematics Gr 6			Platinum mathematics Gr 6			Headstart mathematics Gr 6			
		Word problem exercises examples		Worked-out examples	Word problem exercises examples		Worked-out examples	Word problem exercises examples		Worked-out examples	
		n	%		*N*	%		*n*	%	*N*	%
Procedural fluency	PF^{+}	6	13	0	8	11	0	7	13	0	-
	PF^{++}	7	15	0	29	32	0	16	29	1	2
	PF^{+++}	12	26	0	3	3	0	4	7	0	-
	PF^{++++}	0	-	0	1	1	0	0	-	0	-
Strategic Competence	SC	0	-	0	0	-	0	0	-	0	-
Adaptive Reasoning	AR	0	-	0	4	5	0	1	2	0	-
Productive Disposition	PD^{+}	1	2	0	2	5	0	0	-	0	-
	PD^{++}	19	40	0	36	41	0	27	49	1	2
	PD^{+++}	2	4	0	2	2	0	0	-	0	-

Note: What follows is a presentation of the analysis of each textbook in terms of their potential to promote each of the strands of mathematical proficiency. It must be noticed that the aim of this study is not to endorse any of the textbooks' approach or set of examples as superior. Rather, the study is intended to investigate the affordances for learning fraction word problems in the three sampled and commonly used Grade 6 CAPS-approved textbooks.

PF, procedural fluency; SC, strategic competence; AR, adaptive reasoning; PD, productive disposition; Gr, grade.

+, 1st degree; ++, 2nd degree; +++, 3rd degree; ++++, 4th degree.

because mathematical language does not come naturally to learners. It needs to be learnt and developed.

PF^{+++}: A total of 26% of the fraction word problems in the Premier textbook promote procedural fluency to the 3rd degree (*PF*$^{+++}$), and 7% in the Headstart textbook. Fraction word problem exercise examples in the Platinum textbook are fewer compared to the other two textbooks, with only 3% of the fraction word problem exercise examples under this category. As a result, Premier affords learners more opportunities to understand technical language in fraction problem solving. The high exposure in the Premier textbook of exercise examples in this category offers learners the opportunity to move back and forth between everyday language and mathematical language. The integration of both everyday language and mathematical language generally promotes mathematical thinking and increases learners' understanding in the mathematics discipline (Patkin 2011). The Premier textbook offers a gradual increase in the cognitive demands of the exercise examples and the development of knowledge. Compared to Platinum and Headstart, Premier exposes learners to more opportunities to enhance their procedural fluency through word problem exercise examples with both everyday English and mathematical verbal cues.

PF^{++++}: All non-routine exercise examples that appear for the second time promote the development of procedural fluency to the 4th degree. There were no fraction word problem exercise examples that promoted procedural fluency in the 4th degree in Premier and Headstart textbooks. Compared to Premier and Headstart, Platinum has only one fraction exercise example that promotes procedural fluency to the 4th degree. One exercise example is not adequate exposure to offer learners the skill sets and cognitive tools to solve problems more flexibly and more accurately. Furthermore, the general absence of *PF*$^{++++}$ in both fraction word problem work-out examples and exercise examples indicates failure to promote strategic competence, to which we now turn.

Strategic competence

Table 2 shows no evidence of Strategic Competence across all three textbooks. Strategic competence consists of mathematising problems through representation and establishing appropriate strategies for the problem (Kilpatrick et al. 2001). In the context of fraction word problems, Copur-Gencturk and Doleck (2021) suggest that multistep word problems that are non-routine foster strategic competence. The absence of strategic competence fraction word problem exercise examples limits learners' exploration of a range of fractional representations that cannot be developed through routine exercises. These kinds of fraction word problem exercise examples are crucial for fraction proficiency because they help expose learners to cognitively varying problems, prompting critical thinking and developing proficiency in higher-order problems.

The absence of examples that promote strategic competence is a concern because there is an increase in demand for such problem-solving skills in the 21st century (Nguyen et al. 2020) and without non-routine problems in all three textbooks, learners do not get the opportunity to refine their problem-solving skills that prepare them for the real world. The limited exposure and development of strategic competence is a concern, as textbooks are used for instructional purposes and for a subject such as mathematics that is hierarchical in nature, learners need such exposure (Doorman et al. 2007).

Adaptive reasoning

For Adaptive Reasoning, Table 2 reveals that there are no fraction word problem exercise examples in the Premier textbook in this category. Platinum textbook offers some affordance to learners in this strand, with 5% of the fraction word problem exercise examples promoting adaptive reasoning. According to the National Research Council (2000), learners can be taught strategies that include the ability to explain their thinking and develop their metacognition. Headstart has 2% of fraction word problem exercise examples for this strand. The exposure to adaptive reasoning problems is limited in all three textbooks. The three textbooks do not take the metacognitive approach because of the scarcity of fraction word problems that develop adaptive reasoning.

Productive disposition

Table 2 shows that in the Premier textbook, productive disposition is promoted at 46%, 48% in Platinum and 49% in Headstart.

PD^{+}: Our analysis revealed that 5% of the fraction word problem exercise examples promote productive disposition in the first degree in the Platinum textbook. There is only one fraction exercise example that promotes first-degree productive disposition (*PD*$^{+}$) in the Premier textbook. There are no word problem exercise examples in the first degree of productive disposition in the Headstart textbook in this category (*PD*$^{+}$). Both Premier and Platinum offer learners minimal opportunities for a more realistic mathematics approach, which, according to Haji, Yumiati and Zamzaali (2019), makes learners aware of the uses of mathematics and how it connects to the real world as they know and interact with it. Premier textbook offers one word problem exercise example that makes use of authentic data and is relatable to learners, making problem solving to be viewed as worthwhile. However, exposure to one-word problem is not sufficient to help learners understand fraction concepts from an everyday experience perspective.

PD^{++}: In Premier Mathematics, 40% of the fraction word problem exercise examples (19 exercise examples) promote productive disposition to the 2nd degree. Compared to Platinum, 41% of the fraction exercise examples (36 in total) in the Premier textbook promote 2nd degree productive disposition.

Headstart textbook offers learners more opportunities to develop productive disposition at the 2nd degree compared to Premier and Platinum, with 49% of its fraction word problem exercise examples promoting PD^{++}. Given that productive disposition is defined as a 'habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy' (Kilpatrick et al. 2001:116), Headstart exposes learners to more scenarios that may be deemed as worthwhile and useful because of their realistic approach. The high exposure of PD^{++} in Headstart affords learners the opportunity to develop competency in mathematical modelling and enables learners to use models as applications to help them solve real-world problems.

PD^{+++}: The fraction word problem exercise examples under this category were categorised by the use of exaggerated everyday scenarios for the purpose of learning mathematical concepts. Table 2 shows that in the Headstart textbook, there were no fraction word problem exercise examples under this category. The absence of this aspect of productive disposition in Headstart limits the potential for mathematical thinking, development of metacognition and high-level thinking skills that are acquired through PD^{+++}.

Conclusion

A close inspection of the findings reveals a scarcity of worked-out examples, with only one worked-out example across the three textbooks for PF^{+} and PD^{++}. Users of the three textbooks are therefore given limited opportunities to learn from worked-out examples as examples are 'shown to contain enough information about the procedures to permit diligent students to learn them without additional instruction' (Zhu & Simon 1987:138).

Learners struggle with fractions and have a limited fraction proficiency in primary school (Roesslein & Codding 2018), and the distribution of these strands in the three analysed textbooks has the potential to reproduce learners who struggle with the concept of fraction word problem solving, simply because they do not offer opportunities to explore challenging problem solving. A study (Turner et al. 2003) that analysed the influence of teacher discourse and student behaviour in mathematics classrooms revealed that Grade 6 learners who were challenged by their teachers in classrooms showed more adaptive strategies compared to those who avoided the challenge completely. In the context of this research, learners equally struggle with challenging problems during teaching and learning from observation because the textbooks they get to independently use do not challenge them to a degree of proficiency and comfortability in solving non-routine problems.

The findings, underpinned by Kilpatrick et al.'s (2001) strands of mathematical proficiency, have revealed that the textbooks commonly used in the Grade 6 educational setting under investigation afford learners limited opportunities to develop their fraction proficiency because of the gaps and focus of textbooks. The development of procedural fluency and productive dispositions is essential for mathematical learning, but this should not be to the detriment of the other strands. This is a concern because these commonly used textbooks are not simply used as workbooks for practice, but are a main teaching and learning resource. Adaptive reasoning and strategic competence were completely neglected altogether in the textbooks. The insufficient opportunities in all three textbooks limit learners' development of the other strands of mathematical proficiency. As South Africa edges towards a new curriculum that will necessitate the publication of new (mathematics) textbooks, our findings are vital in informing the content creation of examples that go into these textbooks.

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Authors' contributions

D.S. produced the first draft; Both D.S. and A.A.E. reconceptualised the data analysis and methodology after the original draft was produced; Both A.A.E. and D.S. conceptualised the methodological approach and contributed to the discussions of the findings section of the article.

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Data availability

The data that support the findings of this study are available from the corresponding author, A.A.E. upon reasonable request.

Disclaimer

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