Optimal design of intermittently operated subsonic-supersonic ejectors

D.vV. Pienaar[1]

(First received February 2002; Final version May 2002)

The optimal design for maximum run duration of intermittently operated subsonic-supersonic air-air ejector pumps of fixed geometry was investigated using an inviscid one-dimensional model of the mixing chamber. The study was confined to the case of matched static pressures of the primary and secondary air streams. An empirical correction factor was applied to the ID results in order to take approximate account of the two-dimensional velocity distributions at the inlet and outlet of the mixing chamber and of viscous shear forces. Optimisation of both the primary and secondary Mach numbers was found to yield significantly longer run durations than when the secondary is flow is sonic and just the primary Mach number is optimised. The most pronounced advantages of optimising both the primary and secondary Mach numbers were obtained in the range of high ejector performance. The main advantage is that an optimised ejector is physically smaller, and requires a smaller storage vessel than one that is not optimised.

NOMENCLATURE

A	area
c	a sub-expression to simplify a long formula
D	diameter
L	length
m	mass of stored air
M	Mach number
P	pressure
P_0	stagnation pressure
R	gas constant
R_E	empirical correction factor of static pressure ratio P_m/P_s
T	temperature
T_0	stagnation temperature
t	run duration
V	storage capacity
W	mass flow rate
γ	specific heat ratio
τ	dimensionless coefficient of run duration

Subscripts

a	ambient conditions
b	outlet conditions of subsonic diffusor
c	corrected conditions after mixing
f	final conditions in storage vessel
i	initial conditions in storage vessel
m	mixed
p	primary
s	secondary
$spec$	specified

INTRODUCTION

Supersonic ejector pumps have been intensively studied for many years by many researchers, for example, Dutton *et al.*,[1] who conducted a thorough theoretical and experimental investigation of a range of supersonic-supersonic air-air ejectors. Optimal supersonic ejector designs, however, seem not to have received that attention. The need for a procedure by which ejector designs can be optimised formally arose from a number of practical ejector applications. These include, amongst others, the ejector pump used in the hot air wind tunnel described by Hattingh *et al.*[2] and the one used in the air bag system designed for the cushioning of the ground impact of a parachute's pay load mentioned by Pienaar.[3] The present investigation was focused on the optimal design of subsonic-supersonic air-air ejectors with matched static pressures of the primary and secondary air streams. The theoretical model used, following Dutton *et al.*,[1] was based on one-dimensional gas dynamic theory. This is admittedly a very simple theory for a complex phenomenon such as the supersonic mixing of two air streams, but, judging by the experimental results from these researchers, it is accurate enough for design purposes. The alternative is to take viscous effects into account in a two-dimensional theory that can trap the shock patterns in the mixing chamber. This could, however, be prohibitively time-consuming in an analysis that is aimed at revealing basic optimisation trends for design purposes, instead of predicting the flow field and shock patterns precisely.

An optimal design procedure was formulated by Dutton & Carroll[4,5] for ejector applications in which the primary mass flow rate, W_p, or primary stagnation pressure, P_{0p}, needs to be minimised. The intermittently operated ejector, however, must be optimised for maximum run duration, which is not necessarily obtained with either minimum W_p or minimum P_{0p}. The purpose of the investigation reported here was to show that optimal configurations exist for this case too, to formulate and implement an op-

1. Senior lecturer, Department of Mechanical Engineering, University of Stellenbosch, Private Bag X1, Matieland, 7602 South Africa

timisation procedure and, finally, to show that significant reductions in run duration are suffered when the selected values of the ejector design parameters M_p and M_s are far from optimal. The optimisation procedure essentially involves a numerical search for the particular combination of the values of M_p and M_s that yields the maximum run duration.

Design features and operating principle

The main components of a subsonic-supersonic ejector pump are the plenum chamber, the primary nozzle, the secondary nozzle, the mixing chamber and the subsonic diffusor. A supersonic jet of high stagnation pressure enters the mixing chamber through the primary nozzle, inducing a secondary fluid of low stagnation pressure to flow at a high subsonic or even sonic Mach number into the mixing chamber. The velocity at the exit plane of the mixing chamber of a properly designed ejector is usually so high that a subsonic diffusor for the recovery of dynamic pressure is indispensable. An example with an annular primary nozzle and a central secondary nozzle is schematically illustrated with its supply system in Fig. 1. Its operating principle is the transfer of momentum from the primary (pumping) fluid to the secondary (pumped) fluid by the action of viscous shear. This induces the secondary fluid to flow from the region of low stagnation pressure in the exit plane of the secondary nozzle into the region of intermediate stagnation pressure in the exit plane of the mixing chamber.

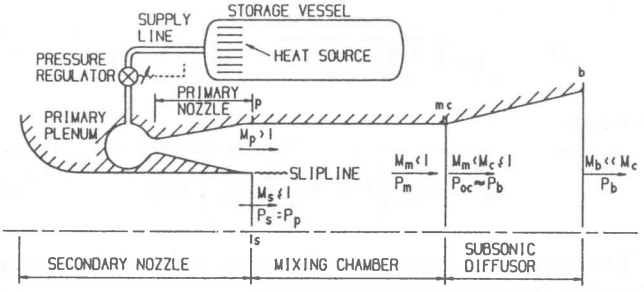

Fig. 1. Schematic representation of ejector system

Intermittent operation

The storage vessel of the ejector system in Fig. 1 is equipped with a capacitive (passive) heat source that limits the decrease in stagnation temperature during discharge. Not much heat transfer from the walls of the storage vessel and supply line occurs during discharge, provided that the L/D ratios of the vessel and supply line are not very large. The stagnation temperature upstream of the heat source will then decrease approximately as if the expansion process were isentropic. This decrease, however, is largely compensated for by the passive heat source at the downstream end of the storage vessel, which heats up the passing air stream again, providing an air supply of nearly constant stagnation temperature, T_{0p}. The regulating valve maintains a constant primary stagnation pressure, P_{0p}, thus compensating for the decreasing vessel pressure, and along with the nearly constant P_{0p}, this yields a nearly constant W_p.

Theory

The viscous shear forces between the two jets entering the mixing chamber (and the resulting momentum transfer) arise from the large velocity gradients of the two-dimensional velocity distribution in the entrance region of the mixing region. However, fairly accurate predictions of ejector performance can be made by a one-dimensional flow model in which the principles of conservation of mass, momentum, and energy are applied to the constant area mixing chamber. The main assumptions of the one-dimensional flow model of compressible mixing at constant area are that:

a) the velocity distributions in the exit planes of the primary and secondary nozzles, the mixing chamber and the subsonic diffusor are uniform,

b) deviations from these uniform velocity profiles, along with viscous wall shear forces in the mixing chamber can be accounted for to fair approximation, as was shown by Dutton *et al.*[1] by applying an empirical correction factor to the pressure recovery ratio P_m/P_s and that

c) the flow through the mixing chamber and subsonic diffusor can be regarded as adiabatic by virtue of insufficient time for significant heat transfer to or from the duct walls.

The stagnation pressure loss in the subsonic diffusor downstream of the mixing chamber is assumed to be negligible compared to the shock losses in the mixing chamber.

Supply system

Assuming that the expansion in the storage vessel upstream of the heat source is adiabatic and isentropic, and that W_p is constant, it follows, as shown in Appendix A, that the maximum available run duration obtained with an initial mass m_i of stored air is:

$$t = \frac{m_i}{W_p}\left[1 - \left(\frac{P_f}{P_i}\right)^{1/\gamma}\right] \tag{1}$$

The run duration is non-dimensionalised with respect to the time $t_{\max} = m_i/W_s$ that it would have taken to evacuate the storage vessel completely, had its rate of mass discharge been equal to the secondary mass flow rate, W_s. This is done merely so that the storage capacity V need not be known *a priori*. The resulting *coefficient of run duration*:

$$\tau = \left(\frac{W_p}{W_s}\right)^{-1}\left[1 - \left(\frac{P_f}{P_i}\right)^{1/\gamma}\right] \tag{2}$$

is a suitable *dimensionless* object function for the optimisation procedure, because its maximum implies that the *dimensional* run duration is a maximum for given secondary flow conditions. The expansion ratio P_f/P_i in the storage vessel (and therefore the available run duration for given

W_p/W_s) is clearly related to the initial pressure ratio P_i/P_a of the supply system, the pressure loss ratio P_f/P_{0p} of the supply line and the required primary stagnation pressure ratio, P_{0p}/P_a, by:

$$\frac{P_f}{P_i} = \frac{P_f}{P_{0p}} \frac{P_{0p}}{P_a} \left(\frac{P_i}{P_a}\right)^{-1} \tag{3}$$

Note that P_f/P_{0p} is a design constant of the supply line, one that is determined uniquely by the layout and dimensions of the supply line, the regulating valve, the plenum chamber and the primary nozzle, whereas the ratio P_i/P_a is a design constant of the storage vessel and its compressor system. The reason for constant P_f/P_{0p} is threefold. Firstly, according to Colebrook,[6] the influence of varying Reynolds number on pressure losses in pipes is negligible in the hydraulically rough region of the Moody chart. Secondly, a design study of Pienaar[7] showed that the flow in typical wind tunnel supply lines falls in this region. And thirdly, the regulating valve can be assumed to be choked when fully open in the last instant of a run, which is not an unduly restrictive assumption. A value of $P_f/P_{0p} = 2$ has been found by Pienaar[7] to make adequate provision for the losses in supply lines of reasonable L/D ratio if its regulating valve is properly selected.

The value of P_{0p}/P_a, on the other hand, and the value of W_p/W_s, must be determined by solving the gas dynamic equations that model the ejector. The solution procedure is based on the theoretical formulations that follow.

Inviscid compressible mixing at constant area

The one-dimensional flow model of inviscid, compressible mixing at constant area as expounded by Dutton *et al.*[1] can be reduced, for the case of identical primary and secondary fluids considered here, to the form:

$$\frac{W_p}{W_s} = \frac{P_p}{P_s} \frac{A_p}{A_s} \frac{M_p}{M_s} \left(\frac{1 + \frac{\gamma-1}{2} M_p^2}{1 + \frac{\gamma-1}{2} M_s^2}\right)^{1/2} \tag{4}$$

and

$$M_m = \left[\begin{matrix} \left(1 - 2\gamma c_m \pm \sqrt{1 - 2\gamma c_m - 2c_m}\right) / \\ \left(2\gamma^2 c_m - \gamma + 1\right) \end{matrix} \right]^{1/2} \tag{5}$$

in which

$$c_m = \frac{\left(1 + \frac{w_p}{W_s}\right)^2 M_s^2 \left(1 + \frac{\gamma-1}{2} M_s^2\right) /}{\left[1 + \gamma M_s^2 \frac{P_p}{P_s} \frac{A_p}{A_s} \left(1 + \gamma M_s^2\right)^2\right]^2} \tag{6}$$

The negative inner root in equation (5) yields a real subsonic solution for M_m, whereas the positive root yields a real supersonic solution, provided that $1 - 2\gamma c_m - 2c_m$ is positive.

Primary to secondary static pressure ratio

Preliminary results of the present investigation have shown that the longest run duration is obtained with ejectors designed for the highest possible primary to secondary static pressure ratio, P_p/P_s. It was decided, however, to restrict the optimisation to the case of $P_p/P_s = 1$, partly because the one-dimensional model of the subsonic-supersonic ejector can reasonably be expected to be fairly reliable in this case, but also because operation near this matched static pressure point was shown by Dutton *et al.*[1] to improve ejector pressure recovery. Another motivation is that this yields a conservative design, in the sense that, according to experimental results of these authors, even better performance can be expected if one were to increase the primary stagnation pressure of a given ejector slightly in order to obtain a value of P_p/P_s slightly higher than unity. This pressure ratio, however, must not become so high that a significant risk of aerodynamic choking in the mixing chamber or of a reverse flow region in the secondary stream arises.

Recovery ratio and run duration

The wall shear force in the mixing chamber and non-uniformities of inlet and outlet velocity distributions are approximately accounted for by an empirical correction factor $R_E = 0.8$, based on the experimental results of Dutton *et al.*[1] This factor represents the ratio of measured P_m/P_s to its theoretically predicted value. The experiments of these authors were conducted, admittedly, with *supersonic-supersonic* ejectors, but it does not seem unreasonable to use the same value of R_E in a design analysis of *supersonic-to-subsonic* ejectors too. One should, however, perform an experimental check of its applicability to supersonic-to-subsonic ejectors too.

Following Dutton & Carroll[4,5] the *corrected* Mach number after mixing is found from:

$$M_c = \left[\left(\sqrt{1 + 2\gamma c_c - 2c_c} - 1\right) / (\gamma - 1)\right]^{1/2} \tag{7}$$

in which:

$$c_c = M_m^2 \left(1 + \frac{\gamma - 1}{2} M_m^2\right) / R_E^2 \tag{8}$$

The corrected *static* recompression ratio of the mixing chamber then follows directly from:

$$\frac{P_c}{P_s} = \begin{matrix} \left(1 + \frac{W_p}{W_s}\right)\left(1 + \frac{A_p}{A_s}\right)^{-1} \\ \times \frac{M_s}{M_c} \left(\frac{1 + \frac{\gamma-1}{2} M_s^2}{1 + \frac{\gamma-1}{2} M_c^2}\right) \end{matrix} \tag{9}$$

For the purpose of the current investigation, however, the *stagnation* recompression ratio:

$$\frac{P_b}{P_{0s}} \approx \frac{P_{0c}}{P_{0s}} = \frac{P_c}{P_s} \left(\frac{1 + \frac{\gamma-1}{2} M_c^2}{1 + \frac{\gamma-1}{2} M_s^2}\right)^{\frac{\gamma}{\gamma-1}} \tag{10}$$

is regarded as the more appropriate measure of ejector performance (or load). It is, after all, a low secondary *stagnation* pressure that really loads an ejector, not a low *static*

pressure, and the ratio P_c/P_s implicitly disregards the substantial contribution of a high secondary dynamic pressure to the overall recovery ratio, whereas the alternative ratio P_c/P_{0s} would disregard the important advantage of subsonic diffusion after mixing.

The required primary stagnation pressure ratio P_{0p}/P_a in equation (3) may now be expressed in terms of the recovery ratio by:

$$\frac{P_{0p}}{P_a} = \frac{P_{0p}}{P_{0s}} \left(\frac{P_b}{P_{0s}} \right)^{-1} \frac{P_b}{P_a} \tag{11}$$

in which

$$\frac{P_{0p}}{P_{0s}} = \frac{P_p}{P_s} \left(\frac{1 + \frac{\gamma-1}{2} M_p^2}{1 + \frac{\gamma-1}{2} M_s^2} \right)^{\frac{\gamma}{\gamma-1}} \tag{12}$$

Therefore the coefficient of run duration can finally be computed using equations (2) to (12).

Computational procedures

The preceding theoretical formulation can be used in various computational procedures, depending on the problem at hand. An *ejector performance calculation*, firstly, involves the direct computation of the recompression ratio, P_{0c}/P_{0s}, once the parameters γ, P_i/P_a, P_f/P_{0p}, P_b/P_a, R_E, the operating point $P_p/P_s = 1$, suitable primary and secondary Mach numbers, M_p and M_s, and the primary to secondary ratio of nozzle exit plane areas, A_p/A_s, have been specified. Such a calculation can be performed explicitly by solving equations (4), (6), (5), (8), (7), (9), (10), (12), and (11) in this order, after which τ follows from equations (3) and (2).

An *ejector design calculation*, secondly, requires the computation of M_p that, for given values of M_s and A_p/A_s, will yield some given recompression ratio P_{0c}/P_{0s}. This is accomplished by estimating the value of M_p initially, which will generally not yield the required P_{0c}/P_{0s} directly, and then finding the actually required value of M_p iteratively, by repeated performance calculations, each with an improved estimation of M_p. (Alternatively one could select M_p and then compute the required value of A_p/A_s iteratively.) This effectively constitutes the iterative zeroing of a scalar function $P_{0c}/P_{0s} - (P_{0c}/P_{0s})_{spec} = f_1(M_p)$.

Thirdly, a *constrained optimal design calculation*, i.e. constrained to a preselected value of M_s, involves not only the computation of the value of M_p that would yield a specified value of P_{0c}/P_{0s} for given M_s in an *inner iteration loop*, but also the computation, in an *outer iteration loop*, of the value of A_p/A_s with which this is achieved *optimally*, in other words with the *maximum possible run duration*. This effectively constitutes the maximization of a function of the form $\tau = f_2(A_p/A_s)$, a function in the computation of which the zero point of said function $P_{0c}/P_{0s} - (P_{0c}/P_{0s})_{spec} = f_1(M_p)$ must clearly be found at each iteration step.

Finally, an *unconstrained optimal design calculation*, i.e. one in which M_s is optimised too (other design considerations permitting), involves the same *inner iteration loop* as in the previous procedure, but now it is the *combination* of A_p/A_s *and* M_s with which this is achieved optimally that must be found in the *outer iteration loop*. This effectively constitutes the maximization of a function of the form $\tau = f_3(M_s/M_p)$, one that also requires the function $P_{0c}/P_{0s} - (P_{0c}/P_{0s})_{spec} = f_1(M_p)$ to be zeroed at each iteration step.

If one experiments numerically with a representative number of combinations of ejector design parameters, then an ejector design that is probably not very far from optimal can be obtained, as shown by Hattingh *et al.*[2] The present investigation, however, supplies a formal procedure for obtaining an optimised design.

Discussion of results

These numerical procedures were applied to an air-air ejector with $P_b/P_s = 1$, $P_{0c}/P_b \approx 1$, $P_b/P_a = 1$ and $R_E = 0.8$, and of which the air supply system is specified by $\gamma = 1.4$, $P_i/P_a = 40$, $P_f/P_{0p} = 2$. Selected optimisation results for this case are plotted against P_{0c}/P_s in Figs. 2-7 and discussed below.

Existence of an optimal secondary Mach number

In the limit of $M_s = 0$ only an infinitely large A_s can accommodate the secondary mass flow rate required in a given application. However, a finite *ratio* of A_p/A_s is needed to ensure that the required P_{0c}/P_{0s} is obtained, which implies that an infinitely large A_p is needed. This, in turn, will require an infinitely large W_p and, consequently, zero run duration will be obtained with a vessel of finite storage capacity. Consideration of the limit of infinitely *large* M_s (allowing for supersonic M_s in this argument), along with the implied (diverging) secondary nozzle shows that it too would require an infinitely large A_s. This line of thought again leads to the conclusion that zero run duration will be available. It is therefore not surprising that an optimal M_s must exist, other design parameters permitting.

The optimal value of M_s is seen in Fig. 2 to be high subsonic throughout the selected range of P_{0c}/P_{0s} which is in *qualitative* agreement with the preceding inductive argument against very low subsonic or very high supersonic M_s. The reason why the optimum values of M_s are relatively close to unity over such a wide range of P_{0c}/P_{0s} as shown in Fig. 2, however, is not so clear. One may be surprised, at first, by the implication that a subsonic diffusor upstream of the secondary inlet of the mixing chamber is undesirable, even though an advantage is apparently to be gained from the higher P_s afforded by subsonic pressure recovery. The explanation for this, however, is clearly given by the argument against very low M_s presented here.

Existence of an optimal primary Mach number

For a given recovery ratio P_{0c}/P_{0s}, the required value of A_p/A_s increases as the selected value of M_p decreases, because low M_p is evidently associated with low pumping capacity, and high A_p/A_s with high pumping capacity. A lower limit on M_p, for which only an infinitely large value of A_p/A_s will yield the required recovery ratio, can be shown

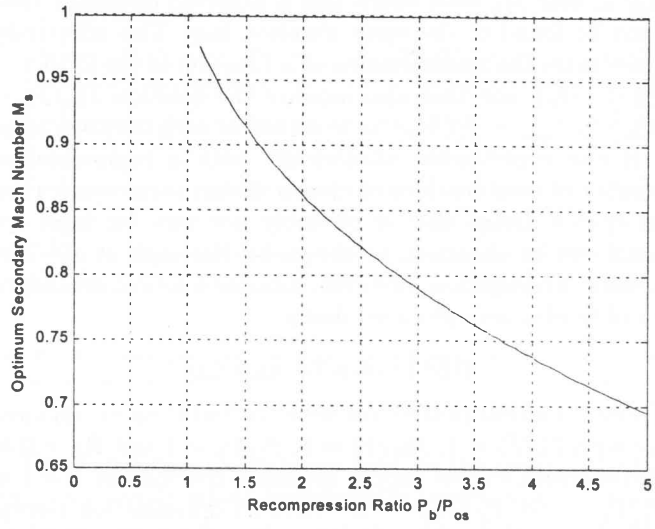

Fig. 2. Decrease in optimal secondary Mach number with increased recompression ratio

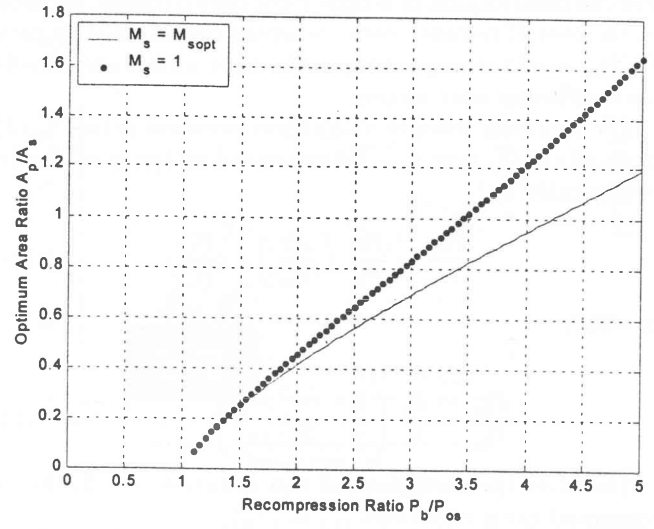

Fig. 4. Decrease in primary area A_p due to optimisation of secondary Mach number

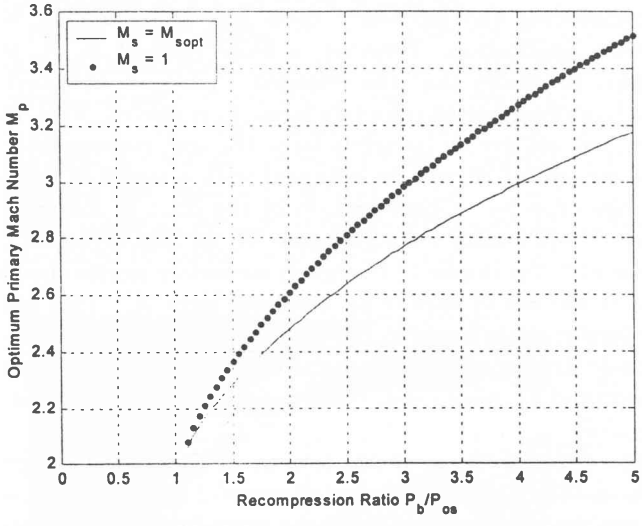

Fig. 3. Decrease in primary Mach number due to optimisation of secondary Mach number

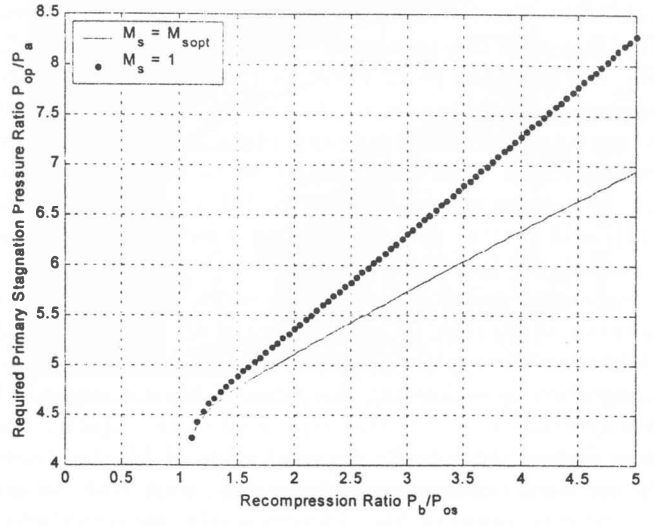

Fig. 5. Decrease in primary stagnation pressure due to optimisation of secondary Mach number

to exist, again implying an infinitely large W_p and zero run duration. In addition, the starting requirement of the supersonic primary nozzle places an upper limit on M_p, one at which the required primary stagnation pressure for starting the nozzle will be equal to the storage pressure less the minimum supply line losses (i.e. with fully opened regulating valve), again reducing the available run duration to zero. Provided that this upper limit of M_p exceeds its lower limit (which it will *not* if the required recovery ratio is higher than allowable by the limitation of the available maximum storage pressure), then an intermediate, optimal value of M_p will exist at which the maximum run duration is available.

The optimal value of M_p is seen in Fig. 3 to be in the intermediate supersonic range which agrees with the qualitative argument above. The largest percentage difference, roughly 38%, between the optimal values of A_p/A_s for $M_s = 1$ and $M_s = M_{sopt}$ shown in Fig. 4 is significantly larger than the largest percentage difference, roughly 20%, between the optimal values of P_{0p}/P_a in Fig. 5. The largest percentage difference between the coefficients of run duration in Fig. 6 is roughly 20%. The largest percentage difference in primary mass flow requirement reflected by W_p/W_s in Fig. 7, however, is only 9%, which is not as much as one would expect at first, judging by the differences of 38% and 20% in A_p/A_s in Fig. 4 and P_{0p}/P_a in Fig. 6, respectively. Careful scrutiny of the mass flow ratio W_p/W_s expressed in terms of stagnation pressure ratio

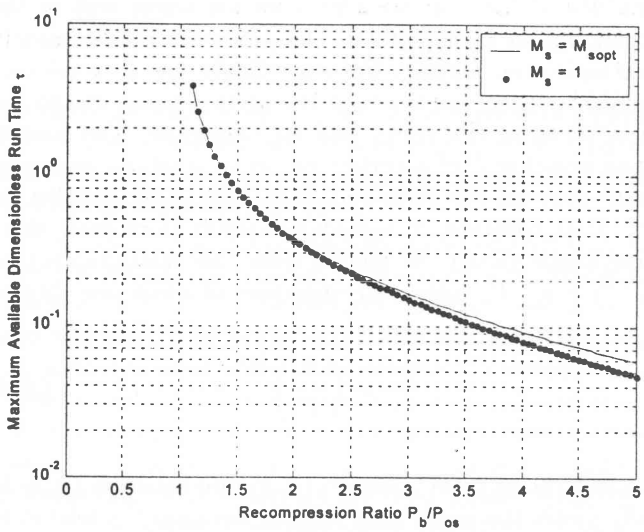

Fig. 6. Increase in run time due to optimisation of secondary Mach number

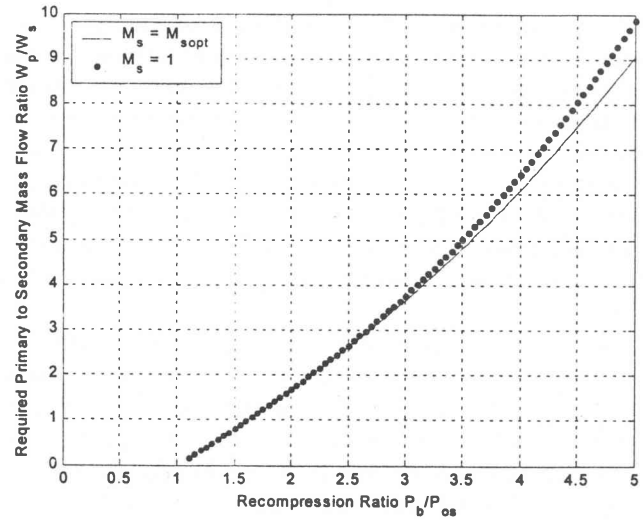

Fig. 7. Decrease in primary mass flow rate W_p due to optimisation of secondary Mach number

$$\frac{W_p}{W_s} = \frac{\frac{P_{0p}}{P_a}\left(\frac{P_{0s}}{P_a}\right)^{-1}\frac{A_p}{A_s}\frac{M_p}{M_s}}{\times\left(\frac{1+\frac{\gamma-1}{2}M_s^2}{1+\frac{\gamma-1}{2}M_p^2}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}} \tag{13}$$

however, reveals that this is because an increase in the value of M_p (in the supersonic range) tends to decrease W_p/W_s, thus opposing the increase in W_p/W_s that is due to the increase in P_{0p} and A_p.

Conclusions

The optimisation of the intermittently operated ejector involves the maximisation of its coefficient of run duration, which is jointly determined by its mass flow ratio and the relevant pressure ratios in equation (2). It should be designed for and operated at the highest expansion ratio, P_p/P_s, at which the required upstream conditions of the secondary flow can prevail. The Mach numbers M_c in the exit plane of the mixing chambers of optimally designed ejectors are significantly lower than the optimal values of M_s, but are still high subsonic, which emphasises the importance of a properly designed subsonic diffusor downstream of the mixing chamber. Subsonic diffusion to low Mach numbers in the secondary inlet of the mixing chamber, on the other hand, reduces the maximum obtainable run durations. Ideally both the primary and secondary nozzles should be designed for the optimal M_p and M_s, at which the maximum run duration for the required recovery ratio is obtained, but if other design considerations preclude the free choice of M_s, then M_p can and should still be optimised for maximum run duration. One should note, however, that even though an ejector with a sonic secondary nozzle (with $M_s = 1$) suffers a significant reduction in run duration compared to the maximum obtainable with an optimum M_s, especially in the range of high recompression ratios, it will be close to optimal in the range of lower recompression ratios.

The results presented in Figs. 2–7 can be regarded as ejector design curves for the particular supply system considered here. This means that, if the required ejector performance P_{0c}/P_{0s} in a specific application is known, one can use these curves to find not only the values of M_s and M_p by which this performance would be *optimally* achieved, but also the corresponding values of A_p/A_s, P_{0p}/P_a and W_p/W_s. The values of M_s, M_p and A_p/A_s allow one to design the primary and secondary ejector nozzles. The values of P_{0p}/P_a and W_p/W_s, on the other hand, along with the specified pressure ratios P_i/P_a and P_f/P_{0p} of a given compressor and supply line, respectively, allow one to find the required storage capacity for a given run duration quite easily by using equations (1) and (2). The higher the required ejector performance, the larger the required ejector and the larger the required capacity of the storage vessel for given run duration will be.

References

[1] Dutton JC, Mikkelsen CD & Addy AL. A Theoretical and Experimental Investigation of the Constant Area, Supersonic-Supersonic Ejector. *AIAA Journal*, **20**, pp.1392–1400, 1982.

[2] Hattingh HV, Pienaar DvV & Bosch LR. A Hot Air Tunnel for Base Bleed Experimentation. *Proceedings of the ICIASF'85 Record*, pp.233–240, 1985.

[3] Pienaar DvV. Enkele Toepassings van Supersoniese Ejektors in die Ontwikkeling en Toetsing van Wapenstelsels. *Proceedings of the Third South African Weapon System Symposium*, Pretoria, Oct. 24-26, Section B, §B3.3, pp.1–17, 1989.

[4] Dutton JC & Carroll BF. Optimal Supersonic Ejector Designs. *Journal of Fluids Engineering, Transactions of the ASME*, **108**, pp.91–93, 1986.

[5] Dutton JC & Carroll BF. Limitations of Ejector Performance due to Exit Choking. *Journal of Fluids Engineering, Transactions of the ASME*, **110**, pp.91–93, 1988.

[6] Colebrook CF. Turbulent Flow in Pipes with Particular Reference to the Transition between the Smooth and Rough Pipe Laws. *Journal of the Institute of Civil Engineers*, pp.133–156, 1939.
[7] Pienaar DvV. The Air Supply System of a Supersonic Blow-down Wind Tunnel. MEng thesis, University of Stellenbosch, 1986.

Appendix A

Conservation of mass in an air storage vessel from which the air flows at a constant mass flow rate, W_p, requires that the difference between the initial mass m_i and the final mass m_f of air in the vessel is related to W_p by

$$m_i - m_f = W_p t \tag{A.1}$$

Solving for t one obtains

$$t = \frac{m_i}{W_p}\left(1 - \frac{m_f}{m_i}\right) \tag{A.2}$$

from which it follows for a vessel of capacity V, on substitution of $m_i = \rho_i\ V$ and $m_f = \rho_f\ V$, that the run time is related to the ratio of final to initial air density in the vessel by

$$t = \frac{m_i}{W_p}\left(1 - \frac{\rho_f}{\rho_i}\right) \tag{A.3}$$

The expansion of the air in the vessel is approximately adiabatic if the heat transfer from the vessel wall to the expanding air is negligible. The process is approximately isentropic too, because the exceedingly low flow velocity inside the vessel implies that the friction losses due to the shear stress at the vessel wall are negligible. The density ρ and pressure P of a perfect gas in an adiabatic and isentropic process, furthermore, are related to one another by $\rho^{-\gamma}\ P =$ constant, a well-known result from basic thermodynamic theory. It follows from this relationship that $\rho_f^{-\gamma} P_f = \rho_i^{-\gamma} P_i$, upon rearrangement of which one obtains

$$\frac{\rho_f}{\rho_i} = \left(\frac{P_f}{P_i}\right)^{1/\gamma} \tag{A.4}$$

Substitution from equation (A.4) into equation (A.3) finally yields the result that the run duration t is related to the ratio of final pressure P_f to initial pressure P_i by

$$t = \frac{m_i}{W_p}\left[1 - \left(\frac{P_f}{P_i}\right)^{1/\gamma}\right] \tag{A.5}$$