Circular Arc/Constant Load Camber Line Equivalence

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During the design process of an axial-flow compressor it is crucial that the correct geometric and aerodynamic properties of the selected blade profiles are used. Failure to do this may result in erroneous blade flow angles at design conditions, premature stall, etc. (overviews of axial-flow compressor aerodynamic design and analysis are given by, among others, Cumpsty [1] and Aungier [2]). It is therefore disconcerting that the relationship between the geometric and aerodynamic properties of the widely used NACA-65 family of blade profiles appears to be misrepresented in the authoritative literature.

The original camber line for the NACA-65 family of blade profiles was the camber line with constant load distribution, ie, constant pressure difference between suction and pressure surfaces [1, p.478]. Later it became common practice to use the circular arc camber line for the NACA-65 family of blade profiles [1, p.479]. Now, since the so-called design lift coefficient determines the maximum camber of the blade section, the equivalent circular arc camber line (for equal design lift coefficient) has to be established when employing such a camber line rather than a constant load camber line. The purpose of this note is to identify the discrepancy that evidently exists in determining this equivalence, and to provide the correct relationship.

NOMENCLATURE

A_0, A_1	Fourier coefficients defined by Equations 3-4
c	chord length
C_L	lift coefficient ($= L/\frac{1}{2}\rho V_\infty^2 c$)
C_{Li}	design lift coefficient
x	distance along chord line from the leading edge
	camber line offset from chord line
V_∞	free stream velocity
α	angle of attack
α_i	design angle of attack
β	camber line slope angle
$\Delta\beta$	camber angle ($= \beta_{x=0} - \beta_{x=c}$)
ϵ	maximum camber (as fraction of chord length)
γ	vortex strength distribution
ρ	fluid density
θ	transformation variable defined by Equation 5
ξ	dummy variable (for x)

Analysis

According to thin aerofoil theory (see for example Abbott and Von Doenhoff [3, Chapter 4]), the relationship between the camber line shape and orientation, depicted in Figure 1, and the loading, is given by

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dy}{dx}\right) \tag{1}$$

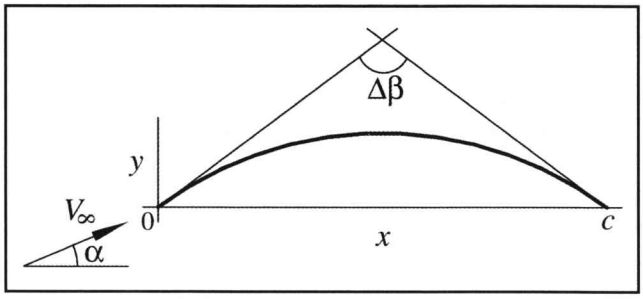

Figure 1: Camber line geometry

Furthermore, the lift coefficient corresponding to a particular camber line obtained from this theory can then be expressed as follows:

$$C_L = 2\pi\left(\alpha - A_0 + \frac{1}{2}A_1\right) \tag{2}$$

where

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{dy}{dx}\,d\theta \tag{3}$$

$$A_1 = \frac{2}{\pi}\int_0^\pi \frac{dy}{dx}\cos\theta\,d\theta \tag{4}$$

In these two expressions, the relationship between x and θ is given by

$$x = \frac{c}{2}(1-\cos\theta) \tag{5}$$

The *design lift coefficient* (also called the *ideal lift coefficient*) is defined as the lift coefficient corresponding to the angle of attack that "avoid infinite velocities at the leading edge" (Abbott and Von Doenhoff [3, p.70]). It can be shown that this criterion is fulfilled if $a = a_i = A_0$, ie, that

$$C_{Li} = \pi A_1 = 2\int_0^\pi \frac{dy}{dx}\cos\theta\,d\theta \tag{6}$$

Constant-load camber line

The shape of this type of camber line is derived by noting that the lift for a constant load distribution $\gamma(\xi)$ = constant is equal to $\rho V_\infty \gamma c$ according to the Kutta-Joukowski theorem. It follows that, for a constant design load distribution $\gamma(\xi) = \frac{1}{2}V_\infty C_L$. The shape of the constant-load camber line can therefore be derived by letting $\gamma(\xi) = \frac{1}{2}V_\infty C_{Li}$ and $\alpha = \alpha_i$ in Equation 1; it then follows that

$$\frac{dy}{dx} = \alpha_i - \frac{C_{Li}}{4\pi}\int_0^c \frac{d\xi}{x-\xi} \tag{7}$$

or, upon evaluating the integral, that the slope of the camber line (when set up at the design lift coefficient) is given by

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$$\frac{dy}{dx} = \alpha_i + \frac{C_{Li}}{4\pi}\left[\ln(c-x) - \ln x\right] \quad (8)$$

This differential equation can be solved, subject to the boundary condition that $y = 0$ at $x = c$. The result is that $\alpha_i = 0$, and that the shape of the camber line is given by

$$\frac{y}{c} = -\frac{C_{Li}}{4\pi}\left[\left(1-\frac{x}{c}\right)\ln\left(1-\frac{x}{c}\right) + \left(\frac{x}{c}\right)\ln\left(\frac{x}{c}\right)\right] \quad (9)$$

The relationship between the maximum camber and design lift coefficient is obtained by noting that the maximum camber occurs at $x = ½\ c$; it therefore follows from Equation 9 that

$$C_{Li} = \frac{4\pi\epsilon}{\ln 2} \quad (10)$$

Also, $\Delta\beta = 180°$ for this type of camber line, as can be deduced from Equation 8. (It should be noted that the definition for design lift coefficient given by Anderson [4, p.250], ie, "the theoretical lift coefficient ... when the angle of attack is such that the slope of the mean camber line at the leading edge is parallel to the freestream velocity" is nonsensical, as borne out by this particular case - a 90° angle of attack would be required according to this definition!)

Circular arc camber line

The shape and slope of a circular arc with radius R are given by

$$\frac{y}{c} = \epsilon - \frac{R}{c} + \left[\left(\frac{R}{c}\right)^2 - \left(\frac{x}{c}-\frac{1}{2}\right)^2\right]^{\frac{1}{2}} \quad (11)$$

$$\frac{dy}{dx} = -\left(\frac{x}{c}-\frac{c}{2}\right)\left[\left(\frac{R}{c}\right)^2 - \left(\frac{x}{c}-\frac{1}{2}\right)^2\right]^{-\frac{1}{2}} \quad (12)$$

Note that the maximum camber here also occurs at the mid-chord position.

The relationship between R and ϵ is obtained by noting that $y = 0$ at $x = 0$ (or $x = 1$); it then follows from Equation 11 that

$$\epsilon = \frac{R}{c} - \left[\left(\frac{R}{c}\right)^2 - \frac{1}{4}\right]^{\frac{1}{2}} \quad (13)$$

from which it can readily be deduced that

$$\frac{R}{c} = \frac{\epsilon}{2} + \frac{1}{8\epsilon} \quad (14)$$

The relationship between the camber angle and the maximum camber can be derived from Equations 12 and 14, the result being identical to that given by Aungier [2, p.66]:

$$\tan\left(\frac{\Delta\beta}{4}\right) = 2\epsilon \quad (15)$$

Combination of Equations 14, 15, 12 and 5 with Equation 6 yields the following integral relationship between the design lift coefficient and the camber angle:

$$C_{Li} = 2\int_0^{\pi} \frac{\cos^2\theta\, d\theta}{\sqrt{\csc^2(\frac{1}{2}\Delta\beta) - \cos^2\theta}} \quad (16)$$

This integral cannot be evaluated analytically; utilising Simpson's first rule of numerical quadrature (see for example Davis and Rabinowitz [5, Chapter 2]) the set of results presented in Table 1 is obtained.

ϵ	$\Delta\beta$ [°]	C_{Li}
0.00	0.0	0.000
0.02	9.2	0.252
0.04	18.3	0.504
0.06	27.4	0.759
0.08	36.4	1.018
0.10	45.2	1.282
0.12	54.0	1.553
0.14	62.6	1.831
0.16	71.0	2.119
0.18	79.2	2.419
0.20	87.2	2.733

Table 1: Design properties for circular camber lines

Parabolic arc camber line

The shape and slope of a parabolic arc camber line which has its maximum camber at the mid-chord position are given by

$$\frac{y}{c} = 4\epsilon\left(\frac{x}{c}\right)\left(1-\frac{x}{c}\right) \quad (17)$$

$$\frac{dy}{dx} = 4\epsilon\left(1-2\frac{x}{c}\right) \quad (18)$$

Substitution of Equation 18 into Equation 6 and evaluating the resultant integral yields the following simple expression for the design lift coefficient:

$$C_{Li} = 4\pi\varepsilon \quad (19)$$

It can also be deduced from Equation 18 that the relationship between the camber angle and the maximum camber is given by

$$\tan(\Delta\beta) = \frac{8\epsilon}{1-16\epsilon^2} \quad (20)$$

Discussion

The design lift coefficients of the three camber lines discussed in the Analysis section are compared graphically in Figure 2. It is evident that there is a close correspondence between the design lift coefficients of the circular arc and parabolic arc camber lines, at least for $0 \leq \epsilon \leq 0.1$; this is not surprising because a parabolic arc is a very good geometrical approximation for a circular arc at moderate camber values.

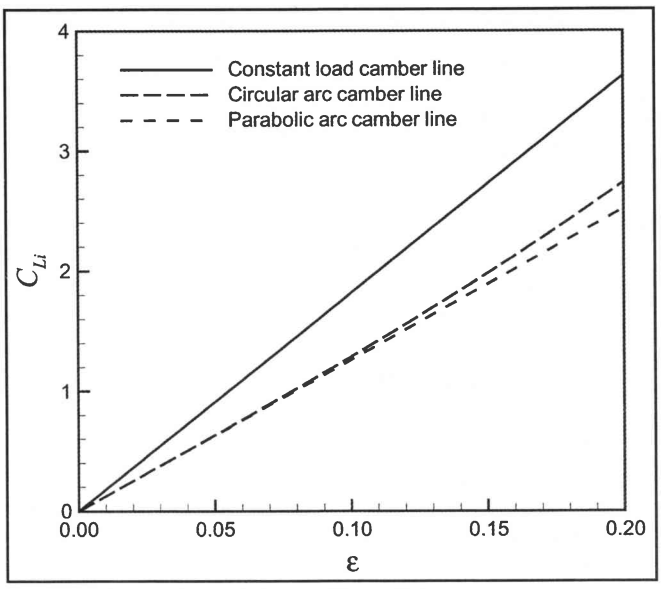

Figure 2: Comparison of design lift coefficients

Also evident is the large difference between the design lift coefficients of the constant-load camber line and the other two camber lines; according to Equations 10 and 19 the design lift

coefficient of a constant-load camber line is larger by the factor $1/\ln 2 \approx 1.44$ than that of the parabolic arc camber line of equal camber (and also, by implication, than that of the circular arc camber line of equal camber at moderate camber values).

The circular arc camber angle that gives a good approximation to the constant-load camber line in terms of *equal design lift coefficient* is given in Table 1. On the other hand, the circular arc camber angle that gives a good approximation to the constant-load camber line in terms of *equal maximum camber* can be found by using Equation 10 of the constant-load camber line and Equation 15 of the circular arc camber line to eliminate ϵ, the result being that

$$\tan\left(\frac{\Delta\beta}{4}\right) = \frac{\ln 2}{2\pi} C_{Li} \approx 0.1103 C_{Li} \qquad (21)$$

These two data sets are compared graphically in Figure 3. On comparison with Figure A2 of Cumpsty [1, p.482], it is clear that the latter data set (that for equal maximum camber) is erroneously purported to be the equal design lift coefficient data set. Note also that Equation 21 is identical to Equation 4-6 given by Aungier [2, p.65] in his discussion on the matter.

Conclusion

The relationship between the camber angle and the lift coefficient for a circular arc camber line that is available in the authoritative literature on axial-flow compressors is clearly erroneous. The reason why this error has not been detected may be that it is compensated for by an equal error "in the opposite direction", ie, the value of ϵ that is used when a compressor blade is manufactured is scaled (inadvertently or consciously) by the factor $1/\ln 2$. In this way the value of $\Delta\beta$ quoted for a particular circular arc camber line is merely used as a code, not as the correct value. But this is pure speculation!

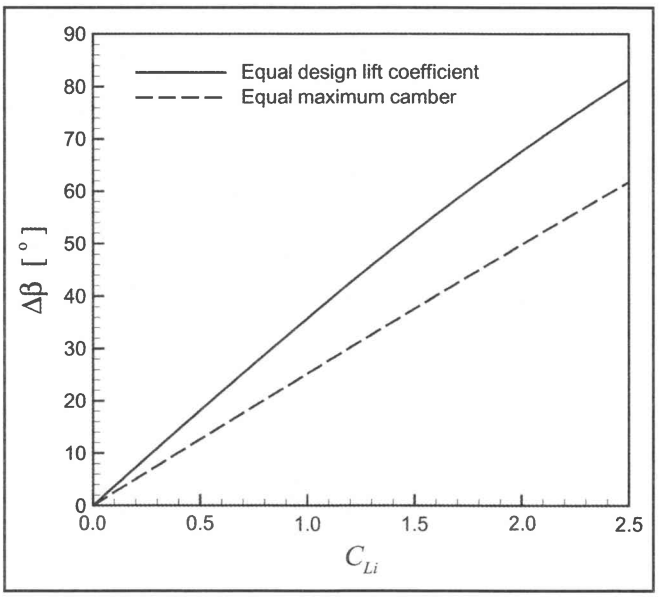

Figure 3: The circular arc camber angle for approximating a constant-load camber line

Reference

[1] NA Cumpsty. Compressor Aerodynamics. *Longman, London, 1989.*

[2] RH Aungier. Axial-Flow Compressors. *ASME, New York, 2003.*

[3] IH Abbott and AE von Doenhoff. Theory of Wing Sections. *Dover, New York, 1959.*

[4] JD Anderson. Fundamentals of Aerodynamics. *McGraw-Hill, New York, second edition, 1991.*

[5] PJ Davis and P Rabinowitz. Methods of Numerical Integration. *Academic Press, New York. 1975.*