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Harnessing GeoGebra as a modelling tool to mitigate undergraduate engineering mathematics students' misconceptions and errors associated with complex numbers

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This article reports on the effectiveness of GeoGebra as a modelling tool to mitigate undergraduate engineering mathematics students' misconceptions and errors associated with complex numbers. GeoGebra is a transformative open-source mathematical software that allows students to visualise and manipulate mathematical objects on different types of digital devices. Despite the centrality of complex numbers in studying vital mathematical concepts such as vectors, eigenvalues, and eigenvectors, studies revealed a prevalence of misconceptions and errors associated with complex numbers. Some students confuse the complex number's representations; others view the representations as autonomous and unrelated. The study adopted a methodological pragmatism research design. It involved volunteering first-year first-semester engineering mathematics students from purposefully selected specialisation groups that included mechanical, industrial, and electrical engineering, at a South African university. The empirical intervention was underpinned by the Realistic Mathematics Education (RME) framework; the data for students' misconceptions and errors were collected from their pre-test and post-test scripts and analysed qualitatively using Donaldson and Orton's errors categories as a lens and quantised or quantitised using a chi-square test. The total frequencies of misconceptions and errors yielded a chi-square statistic of 7.9584 and a p-value of 0.004787, which was statistically significant at $p < 0.05$.

Contribution: The study's key findings strongly suggest that GeoGebra-facilitated intervention effectively mitigates undergraduate engineering mathematics students' total misconceptions and errors associated with complex numbers more than the traditional intervention. This indicates teachers can harness GeoGebra, reducing students' misconceptions and errors associated with complex numbers and improving the quality of teaching and learning complex numbers and tertiary engineering mathematics education.

Keywords: complex numbers; misconceptions; errors; GeoGebra.

Introduction

A complex number is any number that can be written in the rectangular (Cartesian) form: $z = a + bj$.

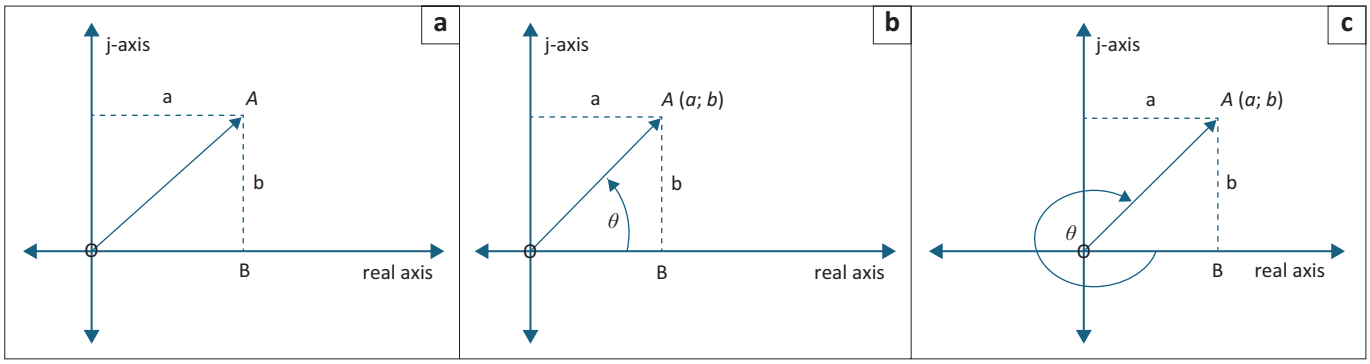
Here, a is the real part, b is the imaginary part, and j is the imaginary unit defined by $j = \sqrt{-1}$, (i.e. $j^2 = -1$). While other researchers and GeoGebra use i this study used j to avoid confusion as i is reserved for electrical current in engineering courses. Given $z = a + bj$, one can plot it on the Cartesian plane and obtain its geometrical representation. The real part is on the horizontal axis, called the real axis, and the imaginary part is on the vertical axis, called the imaginary axis or simply the j-axis (see Figure 1a).

If, in Figure 1a, we let $0A = r$ and θ be angle AOB measured anticlockwise, we obtain Figure 1b. Angle θ can also be obtained by measuring AOB clockwise (see Figure 1c).

According to Bird (2017), from triangle AOB, one can obtain the following trigonometric equations: $a = r \cos \theta$ and $b = r \sin \theta$

Therefore, the rectangular complex number, $z = a + bj$, can be expressed using the following trigonometric equation (see Equation 1) (Bird, 2017):

$$z = r \cos \theta + rj \sin \theta = r (\cos \theta + j \sin \theta) \quad \text{[Eqn 1]}$$



Source: Bird, J. (2017). Higher engineering mathematics (8th ed.). Taylor & Francis Group

FIGURE 1: The geometrical representation and link between the complex number's rectangular and polar forms; (a) is represented geometrically; (b) is (a) if OA is equated with r and angle AOB measured anticlockwise; and (c) is (b) with angle AOB measured anticlockwise.

Equation 1 is called the trigonometric (polar) form of a complex number and is usually abbreviated as seen in Equation 2 (Bird, 2017):

$z = r \angle \theta$, or $z = (r; \theta)$ [Eqn 2]

The angle θ (positive if measured anticlockwise or negative if clockwise) is called the argument. It is measured in either radians or degrees, and r is called the modulus or absolute (Bird, 2017).

It can also be shown using appropriate mathematical procedures that $z = a + bj$ can be written in exponential form as well, viz.: $z = re^{j\theta}$ where θ is only measured in radians (Bird, 2017).

The argument and the modulus are the same for both polar and exponential forms. Hence, the difference between the polar and exponential forms lies in the notations, especially when θ is in radians; otherwise, convert degrees into radians (Bird, 2017). Therefore, the polar and exponential forms have the same geometrical representations. Hence, the complex number has three different algebraic representations: rectangular form ($z = a + bj$), trigonometric (polar) form ($z = r(\cos\theta + j\sin\theta)$ or $z = r \angle \theta = (r;\theta)$) and exponential form ($z = re^{j\theta}$) (Bird, 2017).

Each representation has a pertinent, rectangular, trigonometric, and exponential form, representing different aspects of a complex number, and each has merits and demerits (Bird, 2017; Hui & Lam, 2013; Panaoura et al., 2006; Smith et al., 2019). Therefore, students must know how the representations relate and be able to manipulate each seamlessly and navigate within and between the various representations (Panaoura et al., 2006; Smith et al., 2019). According to the authors, this competence requires deep content knowledge and arithmetic skills in algebraic, trigonometric, and exponential functions. However, from the experience of the authors of this article in teaching complex numbers to undergraduate engineering mathematics students, it has been observed that many of these students' content knowledge and arithmetic skills of these essential functions (algebraic, trigonometric, and exponential) appear to be shaky, although they are covered in Grades 10–12 as required by the Curriculum and Assessment Policy Statement (CAPS) for Mathematics in the Further Education and Training (FET) Phase (Department of Basic Education [DBE], 2011).

A study by Hui and Lam (2013) also revealed that most students need clarification about geometrical and algebraical representations of complex numbers. The misconception was also observed in the study of Panaoura et al. (2006). Most students often approach algebraic and geometric representations of complex numbers from fundamentally different perspectives and consider the two representations separate and autonomous.

It is on this basis that the study explored the affordances of GeoGebra as a mathematical, digital, and technological tool that can facilitate multiple representations (Karakok et al., 2014; Kin, 2018) and meaningfully support the integration of algebraic and geometric representations of complex numbers mitigating undergraduate engineering mathematics students' misconceptions and errors associated with complex numbers.

For Hohenwarter et al. (2008), GeoGebra is freely open and accessible anywhere and anytime, promoting equal access to technological resources, barring data costs and access to devices. GeoGebra-facilitated intervention was buttressed by RME's principles: the guided reinvention, intertwinement, and level.

Research problem

Complex numbers form an integral part of mathematics and are an obligatory content of tertiary mathematical education (Anevska et al., 2014). The Faculty of Science yearbook for undergraduate programmes at a South African university shows that complex numbers constitute around 15% of its first-year first-semester Engineering Mathematics and form an integral part of the study of vectors, eigenvalues, and eigenvectors taught in first-year second-semester Engineering Mathematics.

For any positive real number, $a, \sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a}$. Ahmad and Shahrill (2014) observed that most students struggle to grasp the notion that $\sqrt{-a}$ can be written as $\sqrt{-1} \times \sqrt{a}$, that is, $\sqrt{-a} = j\sqrt{a}$ with $j = \sqrt{-1}$.

Many students need clarification on different representations of complex numbers (Hui & Lam, 2013). Furthermore, the authors observed that understanding one representation does

not guarantee understanding the other. In a similar study, Panaoura et al. (2006) observed that some students consider the complex number's representations separate and autonomous. To fill this void, the study explored and harnessed GeoGebra's visual and multi-representational affordances (Karakok et al., 2014; Kin, 2018). GeoGebra was explicitly used as a mathematical modelling tool to meaningfully link, connect, and work simultaneously and interchangeably with the algebraic and geometric representations of complex numbers, buttressed by the guidance and intertwinement principles of RME. Most students must understand that to simplify certain expressions containing multiple complex numbers' representational forms, they must first identify the most appropriate form and convert all the other representations before performing the relevant procedural and algorithmic operations (Ramaila & Seloane, 2018). Many students might find GeoGebra-enriched illustrations more meaningful. This can enhance their ability to integrate different complex number representations to mitigate misconceptions and errors associated with complex numbers. In addition, GeoGebra-enriched activities can enable students to navigate within and between the various representations of complex numbers to identify, convert comfortably, flexibly, and effectively use the most appropriate form for a given task.

Given this background, this study aimed to investigate the effectiveness of GeoGebra as a modelling tool in mitigating undergraduate engineering mathematics students' misconceptions and errors related to complex numbers.

Purpose of the study

The study investigated whether using GeoGebra as a modelling tool mitigates first-year first-semester undergraduate engineering mathematics students' misconceptions and errors associated with complex numbers than the traditional method. The following objectives underpinned the empirical investigation:

- To classify and analyse undergraduate engineering mathematics students' misconceptions and errors related to complex numbers found on test scripts using Orton and Donaldson's categories as a lens.
- To determine the effectiveness of GeoGebra as a modelling tool in mitigating undergraduate engineering mathematics students' misconceptions and errors associated with complex numbers.

Literature review

Research on the teaching of complex numbers

Despite complex numbers being a vital mathematical topic in post-secondary school, recent studies have yet to be conducted on mitigating misconceptions and associated errors, especially using a digital mathematical tool like GeoGebra in South African tertiary institutions. This poses a considerable challenge to lecturers and students, as lecturers must be aware of these misconceptions and errors and how to design pedagogical interventions to mitigate them.

Ahmad and Shahrill (2014) conducted a study that focused on the complex number's rectangular form, $z = a + bj$, and arithmetic operations, that is, additions, multiplications and divisions on complex numbers, including the role of $j = \sqrt{-1}$. In addition to conceptual difficulties, the study identified students' misconceptions and errors, like $-j^2 25 = -j5$; $-j^2 25 = -j25$ and $j^2 = 0$. Some students did not replace $\sqrt{-1}$ with j; others ignored it, while some ignored its powers. Furthermore, some students generally have algebraic misconceptions, and there was an overreliance on a calculator by some students to find the square root of a negative number. The students cannot understand that $j\sqrt{2} = \sqrt{2}j$ (commutative property), leading the authors to the conclusion that students have misconceptions and cannot understand the notion that $j = \sqrt{-1}$, (i.e. $j^2 = -1$).

Panaoura et al.'s (2006) study examined students' ability to identify more appropriate representations, convert between the representations of complex numbers, and solve problems expressed in different representations. The study showed that students fail to recognise that all these representations represent one entity. These misconceptions hinder the students in identifying, converting, and using the most appropriate representations where needed.

Ramaila and Seloane's (2018) study revealed that only 23 students out of 70 undergraduate engineering mathematics students at a South African university could successfully simplify the multi-representational expression,

$$\frac{2j}{1-\sqrt{-4}} + 3e^{1-j}$$

and leave the answer in rectangular form.

The remaining 47 failed to identify and use the rectangular form in the arithmetic operations. In addition, the study found that students generally struggle with expressions containing multiple representations and arguments measured with various units (radians and degrees). Like Panaoura et al. (2006), Ramaila and Seloane found that students generally make errors and have misconceptions about complex numbers' representational forms. Students lack adequate knowledge of navigating within and between the different representations. In response to this predicament, Ramaila and Seloane recommended implementing innovative remedial interventions to deepen students' understanding of complex numbers.

Smith et al.'s (2019) study investigated seven junior physics students' facilities, difficulties, and strategies they exhibited when solving complex numbers problems that needed either consistent use of one representation or fluent switching between the algebraic and geometric representation.

The authors first taught the students complex numbers, including how to change efficiently between the different

representations of complex numbers where appropriate. According to the authors, identifying and using the most suitable representation and moving fluidly between the complex numbers' representations exhibit 'expert-like behaviours' and require a connected conception of a complex number's algebraic and geometric representations. Worryingly, 57.1% (4 out of 7) of students exhibited difficulties and misconceptions about the different complex number representations. Students lacked a connected conception of the complex numbers' representations, which hindered them from identifying and consistently using or switching to the most appropriate representation. The authors recommended instruction relating to algebraic and geometrical representations, lessening the abstractions of complex numbers.

Norlander and Norlander (2012) investigated how 47 Swedish engineering university students understood complex numbers through a questionnaire. The questionnaire was administered after the students were taught complex numbers for a week and aimed to 'pinpoint the misconceptions and identify the most common difficulties with complex numbers'. The authors further used an identification test administered to another 62 students to check students' conceptions and misconceptions of complex numbers. Based on students' answers to the questionnaire and identification test, Norlander and Norlander classified students' responses to complex numbers into four categories: (1) mathematical artifice; (2) a two-dimensional view; (3) a symbolic view; and (4) the mystery view or an ungraspable mystery.

Students in the mathematical artifice category view complex numbers as manufactured, an artificial artifice, and doubt their legitimacy.

Students in the two-dimensional view conceive a complex number as two separate entities instead of one unified entity comprised of two parts. Students in this category need help comprehending that all real numbers are complex numbers. One student argued that the number j is not a complex number because it has only one part, the imaginary part, and no real part. According to the student, a complex number has two parts, the real and imaginary parts.

Students in the symbolic view category associated a complex number with the imaginary unit j. The majority (84%, 82% and 86%) of students answered that -2.5, $5\cos\pi$, and $\cos\pi + \sin\pi$, are not complex numbers. According to the students, they want to see the imaginary unit j explicitly. This finding is in line with Conner et al.'s (2007) study, which found that some prospective high school teachers' conceptual understanding of complex numbers does not go beyond the symbol j.

Responses in the mystery view or an ungraspable mystery category showed that students attached their attitudes or emotions and (mis)conceived a complex number as difficult, tricky, complicated, and abstract.

A considerable number of students (34%) did not attempt a single questionnaire test item.

Around an eighth (12.8%) of students indicated in the questionnaire that they struggle memorising complex numbers' rules and formulas. Another 12.8% mentioned that they have difficulties managing polar or trigonometric representations. According to Veloo et al. (2015), the more a topic is perceived as difficult, the higher the possibility of students committing errors when working on it. More than one-fifth (21.3%) of students explicitly said that complex numbers are challenging because they can't visualise them, and in fact, they doubt their legitimacy. However, a small percentage (19.1%) of students were very specific about their difficulties.

A mere 11.3% of students could comprehend complex numbers correctly and adequately answered all identification test items.

Given these misconceptions and errors obstructing meaningful learning of complex numbers, Norlander and Norlander (2012) concluded that due to the abstract property of complex numbers, an innovative visual approach that will lessen their abstraction is recommended for teaching complex numbers.

Norlander and Norlander's (2012) study and all the other relevant studies only identified students' misconceptions and errors associated with complex numbers. Still, they didn't explore the GeoGebra-facilitated intervention's efficacy in mitigating them, and this research gap underscored the need for an empirical investigation as per their recommendation.

The use of GeoGebra as a modelling tool in teaching and learning

Many researchers at different educational levels and from different countries observed that GeoGebra positively impacted the teaching and learning of many mathematical topics, including: fractions (Bulut et al., 2016; Kin, 2018; Supriadi et al., 2014); calculus (Oscal, 2017; Zulnaidi & Zamri, 2016); functions (Ogbonnaya & Mushipe, 2020); and complex numbers (Karakok et al., 2014).

Pfeiffer (2017) used GeoGebra to model transformations of functions and circle geometry. According to him, GeoGebra-enriched activities allowed foundation programme students in a South African university to acquire physical and logico-mathematical knowledge, and GeoGebra can model many mathematical topics.

Phan-Yamad and Man (2018) found that GeoGebra helps teach statistical graphs. A study by Tay and Mensah-Wonkyi (2018) revealed that Ghanaian senior high students believed that GeoGebra makes lessons more engaging, practical, and easy to understand, enabling them to become better circle geometry solvers. Esguerra-Prieto et al. (2018) found that GeoGebra's graphical (geometrical) properties of the representations of arithmetic operations are more user-friendly than those of MATLAB for teaching complex numbers and their arithmetic operations.

Karakok et al. (2014) explored the multi-representational affordance of GeoGebra to show the connection of a complex number's form (rectangular, polar) to its geometrical representation. They also explored teachers' conceptual knowledge of various forms of complex numbers and how they switch between them. The results showed a varying conceptual understanding of the different forms of complex numbers depending on the teacher's teaching experience and not a conceptual understanding of complex numbers.

Shadaan and Eu (2013) concluded that GeoGebra allowed 53 nine-year-old Malaysian students to self-discover, inquire, engage physically, interact, and collaborate. Furthermore, Diković (2009) found that GeoGebra helped college undergraduate students gain positive knowledge, investigation, and exploration skills through self-discovery while encouraging interaction, cooperation, and collaboration while learning calculus.

On this basis, the study sought to compare the efficacy of GeoGebra-enriched intervention mainly driven through the interaction, level, guided reinvention, and intertwinement principles of RME with the conventional marker-and-whiteboard, pen-and-paper traditional approach in mitigating students' misconceptions and errors associated with complex numbers.

Misconceptions and errors

According to Riccomini (2005), there are two types of errors: unsystematic and systematic. According to the author, systematic errors usually result from inaccurate, inappropriate, or incomplete mathematical thinking and reasoning. Unsystematic errors are unintended, non-repeating wrong answers and careless mistakes, and can be corrected by the students.

Matlala and Luneta (2018) provided a student's vignette depicting an example of unsystematic error in their article. According to the authors, the student was asked to simplify the exponential expression,

$$\frac{18^x\left(2\cdot 3^{1-x}\right)^2}{2^{x-1}}$$

The student displayed a solid understanding of exponential rules and correctly wrote, as follows:

$$\frac{18^x\left(2\cdot 3^{1-x}\right)^2}{2^{x-1}}=\frac{\left(2\cdot 3^2\right)^x\left(2\cdot 3^{1-x}\right)^2}{2^{x-1}}=\frac{\left(2^x\cdot 3^{2x}\right)\left(2\cdot 3^{2-2x}\right)}{2^{x-1}}$$

However, the student forgot to raise the factor 2 inside the second brackets (extreme right) by the exponent 2. Therefore, the error is unsystematic. Matlala and Luneta (2018, p. 315), consistent with Riccomini's (2005) view, argue that this error type can be corrected by students themselves by 'reflection and looking back after arriving at the solution'. Matlala and Luneta advise teachers to constantly remind their students to look back at their solutions for validation.

However, Barlow et al. (2018) stress that some unsystematic errors are not only confined to arithmetic operations; some result from incorrect or unjustifiable mathematical thinking, answers, and strategies. McNamara and Shaughnessy (2011) concur and caution teachers to not always interpret students' errors as careless mistakes that students can correct. Keeley et al. (2007) posit that some misconceptions could emanate from inside the classroom and caution teachers about the activities, models, representations, words, etc., used during class instruction as they may be interpreted differently by the students, resulting in students' misconceptions.

Rushton (2018) stresses the importance of error identification and views unsystematic errors as calculation or routine procedure mistakes. Systematic errors (misconceptions) are the intended, repeated wrong answers methodically constructed and consistently produced.

Machaba (2016) and Roselizawati et al. (2014) add that misconceptions can arise when students try or fail to construct and organise new knowledge using prior knowledge. Smith et al. (1993) posit that misconceptions can be from over-generalisation from an existing domain (e.g., non-negative real numbers) to a new one (e.g., complex numbers). The assertion by Veith and Bitzenbauer (2021) that their equation (see below) is contradictory is a classic example of a misconception arising from over-generalisation, as seen in Equation 3:

$$1=\sqrt{1}=\sqrt{\left(-1\right)^2}=\sqrt{-1}\cdot\sqrt{-1}=i\cdot i=-1 \qquad \text{[Eqn 3]}$$

The authors used the property (prior knowledge) $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$, valid for non-negative real numbers, to generalise it to construct new knowledge (complex numbers) that $\sqrt{\left(-1\right)^2}=\sqrt{-1}\cdot\sqrt{-1}$. Hence, their equation is invalid, and there is no contradiction, as postulated by the authors. $1=\sqrt{1}=\sqrt{\left(-1\right)^2}\neq\sqrt{-1}\cdot\sqrt{-1}=i\cdot i=-1$.

Kin Eng and Fui Fong (2020) also maintain that these properties are valid in the subset (e.g., real numbers) but are not valid in the corresponding superset (e.g., complex numbers). According to Kin Eng and Fui Fong, these misconceptions arising from the real numbers context (prior knowledge) can hinder meaningful and coherent learning of complex numbers (new knowledge). Kin Eng and Fui Fong recommend implementing instructional interventions to correct such misconceptions. For Keeley (2012), although misconceptions are not necessarily bad, they can be significant barriers to learning and must be corrected. In support of implementing instructional interventions to correct misconceptions, Machaba (2016) argues that students should not just be told they are incorrect.

Instead, they should be exposed to instructional experiences that confront and guide them through conceptual change, allowing them to reorganise and reconstruct their thinking

willingly. That is precisely what this article strived to achieve with a GeoGebra-supported approach.

Luneta (2008, 2015) posits that errors may be symptoms of the difficulties students had during a learning experience, and misconceptions are observable in students' work as errors, indicating that errors are symptoms of misconceptions students hold.

Luneta's (2008, 2015) assertion seems supported by Smith et al.'s (2016) research study, which used four ungraded tests (two announced and two unannounced), homework, and formal mid-term and final examination test items collected sporadically over four years to identify undergraduate physics students' difficulties, errors, and misconceptions associated with complex numbers. Each year, students were taught complex numbers and given relevant homework three weeks before the administration of the first unannounced, ungraded pre test. The unannounced tests were the same and administered as a pre-post test. The pre test was administered a day before, and the post test was administered after a 10-minute review of complex numbers. The announced test was then administered between the unannounced tests. Students' misconceptions, difficulties, and errors were classified into three categories, namely: (1) performing arithmetic calculations; (2) switching between different complex number forms; and (3) selecting the appropriate complex number form for the task.

In the two unannounced tests, students were given two complex numbers, $z_1 = -3 + 7j$ and $z_2 = 3e^{2j}$, and were asked to calculate z_1^2, $|z_1|^2$, z_2^2, and $|z_2|^2$ for each of the two complex numbers.

Recorded students' notable misconceptions, errors, and difficulties were:

- The students have misconceptions and difficulties associated with the imaginary unit *i*. More than 10 per cent (10.5%) (9 out of 86) of students wrote $j^2 = 1$ instead of the correct $j^2 = -1$ in the pre test.
- The authors argue that instructional language (a classic example of misconception emanating from the class) may have led to students' misconceptions. More than 7% (6 out of 86) of students in the pre-test calculated $|z_1|^2$ instead of z_1^2 Students may have confused the two as both are squares. Keeley et al. (2007) cautioned teachers about instructional and terminological language as they can be a potential source of students' misconceptions and, therefore, should anticipate any possible sources of misconception.
- According to the authors, 31.5% (27 out of 86), 7.2% (6 out of 83), and 4.3% (1 out of 23) of students did not respond to z_2^2 in the unannounced pre test, announced test, and unannounced post test, respectively. Again, 61.1% (33 out of 54), 15.7% (13 out of 83) and 0% of students did not respond to $|z_2|^2$ in the unannounced pre test, announced test, and unannounced post test, respectively. The authors suspect it could indicate students' difficulties with exponential functions.

The presumption is consistent with Luneta's (2008, 2015) suggestion that errors may be symptoms of difficulties for students.

In the following paragraphs, we highlight some of the students' noteworthy misconceptions, errors, and difficulties when: (1) converting between complex numbers' different forms in one of the post tests; and (2) selecting the appropriate complex number form for the task in the mid-term and final examination.

According to the authors, out of 23 students, only 47.8% (11 students) managed to convert $3e^{\frac{-i\pi}{4}}$ into rectangular form correctly, 4.3% (1 student) gave no response, and the remaining students displayed difficulties simplifying to the final answer.

More similar persistent errors, misconceptions, and difficulties were observed when students were asked in the post test to convert $z = \sqrt{3} - j$ into exponential form.

Only 26.1% (6 out of 23) of students converted correctly. The remaining 13% (3 out of 23) did not respond and 60.8% (14 out of 23) had difficulties and misconceptions simplifying trigonometric functions (e.g., incorrect sign, quadrant, identity, algebraic errors, etc.)

More than half (52%; 13 out of 25) of the students in the mid-term exam missed the task's context and used the incorrect form to evaluate *ln* (*z*). This is a typical example of structural error. Despite this number being reversed in the final examination to 80% (20 out of 25) and students using the correct form, persistent calculation errors, misconceptions, and difficulties were noticed in students' workings, and they still could not evaluate *ln* (*z*) correctly.

Luneta (2015) emphasises the teachers' knowledge of students' errors and misconceptions; according to him, teachers should provide opportunities for students to display their errors because they are critical stepping stones for effective pedagogical instruction.

According to Luneta (2008, 2015), misconceptions manifest as errors and can be identified from students' workings.

Given this view, this article analysed students' misconceptions and errors associated with complex numbers found on test scripts qualitatively using Orton and Donaldson's errors classification as a lens. Then, the chi-square test was used to quantify misconceptions and errors.

In Orton's (1983) clinical study, researchers interviewed 110 students aged between 16 years and 22 years to investigate their understanding of differentiation and integration. Sixty students were from four high schools, and the remaining 50 were prospective teachers from two colleges. The questionnaire items included elementary algebra, limits and infinity, and integration and area. Based on students' responses, Orton noticed that many questionnaire items were

difficult for students, especially the ones about integration as the limit of the sum. These findings echo other (Ahmad & Shahrill, 2014; Norlander & Norlander, 2012; Ramaila & Seloane, 2018; Smith et al., 2019) studies regarding students' difficulties with complex numbers.

According to Orton (1983), many teachers have accepted students' difficulties and either do not teach integration or teach it socially as a rule (antidifferentiation) without meaning. Orton, in attempting to simplify and generalise students' errors, realised that some responses behaved arbitrarily and failed to take account of the constraints laid down in what was given as too general and sometimes meaningless, and hence modified and used Donaldson's (1963) descriptions of three types of errors to accommodate this category of errors.

The three error groups are; (1) structural (systematic, misconception) errors, which are repeated errors emanating from inaccurate or partially correct conceptual mathematical thinking and stemming from conceptual and procedural knowledge gaps; (2) executive (unsystematic) errors, which are non-repeated mistakes, slips, or blunders that students make during manipulations and can be corrected by the students themselves and (3) arbitrary errors, which behave arbitrarily, outside the set domain, too general and vague, and sometimes meaningless responses.

Orton (1983), along with the authors of this article, acknowledges some overlaps between these error types and doesn't claim that these types are the only ones. However, they invoked them for this study and article.

The article analysed students' test scripts qualitatively, invoking Donaldson's (1963) and Orton's (1983) error classification as a lens to identify and classify undergraduate engineering mathematics students' misconceptions and errors. The frequency of students' misconceptions and errors will indicate whether GeoGebra as a modelling tool mitigates misconceptions and errors associated with complex numbers better than the traditional method.

Research methodology

Research design

This experimental research study followed a pragmatic approach (Colin, 2023; Creswell, 2018). According to Colin (2023), methodological pragmatism does not tie the researcher to any research design (e.g., quantitative, qualitative, or mixed method). Instead, it allows the researcher to use any method from any approach that can further their research goals. Kaushik and Walsh (2019) argue that pragmatism is an appropriate research paradigm, particularly for gathering evidence to determine the effectiveness of interventions. Students' errors were collected from pre-post test scripts, analysed qualitatively, and then quantised or quantitised, and finally, the chi-square test was used for numerical interpretation. According to Sandelowski et al. (2009), quantising converts qualitative data into quantitative data.

The Experimental Group (EG) was exposed to GeoGebra-mediated instruction, while the Control Group (CG) was exposed to traditional instruction to determine which intervention effectively mitigated students' misconceptions and errors.

The duration of the instructional interventions was 610 minutes over two weeks for each group. It included 160 minutes for administering the tests.

An 80-minute test, which served as a pre-post test, was administered to the two groups. The test consisted of two 40-minute sections, named test 1 and test 2 for this study. Test 1 covered the arithmetic operations (addition, subtraction, multiplication, and division) of rectangular complex numbers only, and test 2 combined all the operations on all complex number representations. Test 1 (pre test) was administered to the two groups simultaneously before any lesson on complex numbers.

Due to the availability of the students:

- The test 2 pre-test was administered to the two groups one day apart.
- Each group wrote the post test (test 1 and test 2) in one sitting but two days apart.

Participants

The study was conducted in a South African university and involved 53 volunteering first-year first-semester engineering mathematics students from two (the university had five groups) purposefully selected groups: mechanical and industrial (constitute one group) engineering and electrical engineering. The two groups were chosen to minimise potential contamination. The groups didn't attend any module classes together.

A total of 26 electrical engineering students formed the EG, along with 27 mechanical and industrial engineering students that constituted the CG.

Implementation of the GeoGebra-facilitated instructional intervention

The lecturer-researcher (the first author) was the regular lecturer of the electrical engineering group only; however, for this study, he taught both the EG and the CG complex numbers. The topics on complex numbers were part of the students' normal curriculum, and students were informed that the results from the study were solely for the study and would not contribute to their formal assessments. The lecturer-researcher taught each group for 450 minutes, the CG outside the group's regular timetable. Still, before the introduction of complex numbers by their regular lecturer, one of the two reviewers and moderators, the EG was taught during their normal timetable. The CG intervention was conducted using the conventional marker-and-whiteboard, pen-and-paper traditional approach. As for the EG, the lecturer was the

primary user (the only one with access) of GeoGebra in class. He used it for demonstration, exploration, explanation, engagement, validation (some exercises were given as 'homework', and feedback was provided in the next class session), and visualisation. Hohenwarter and Fuchs (2004) mention this as one of the applications of GeoGebra. Lederman and Niess (2000) add that this technology usage can potentially improve teacher effectiveness and student learning.

Ibrahim and Llyas's (2016) and Ocal's (2017) studies also found this approach compelling. According to Sherman (2014), technology can be used as an amplifier to perform tedious computations quickly, allowing students to focus primarily on observations and developing insight rather than being troubled by manual procedures. Technology can also be used as a reorganiser to extend students' thinking, allowing them to focus on looking for patterns, identifying invariances, making and testing conjectures, and thus giving them access to higher-level processes (Sherman, 2014).

The intervention was further guided by Simon's (1995) Hypothetical Learning Trajectory (HLT), driven mainly by the guided reinvention, level, and intertwinement principles of RME, presented as a mathematical path within six phases (see Table 1).

Phase one was guided conventionally through the reality principle of RME with a whiteboard and marker. According to Drijvers (2012), activities in RME do not necessarily need to be from a realistic context as they can still be experientially actual (personally meaningful mathematical activity). Solutions of quadratic equations like, $ax^2 + bx + c = 0$, $a;b;c \in R$ and $a \neq 0$, are covered in Grades 10–12 (DBE, 2011), and are therefore realistic to the students. Students saw that the real number system (R) is incomplete since it can't solve quadratic equations like $x^2 + 2 = 0$. The emphasis was to help students comprehend the square root of a negative real number. The real number system (R) was then extended to make a complete number system, the complex number system (C).

TABLE 1: Summary of Hypothetical Learning Trajectory for teaching complex numbers.

Phase	Duration (min)	Designed activities	Prominent RME principles
1	20	Solutions of quadratic equations, especially those with non-real roots.	Reality.
2	25	Introduction of a rectangular complex number.	Interaction.
3	135	Modelling the sum, conjugate, and scalar multiples of two rectangular complex numbers. Simplifying complex expressions in rectangular complex number sums, differences, products and quotients.	Interaction, level, guided reinvention.
4	45	Introduction of polar and exponential forms of complex numbers. Linking polar and rectangular forms. Modelling the product or quotient of complex polar numbers. The conversions between the different representational forms of complex numbers.	Guided reinvention, intertwinement.
5	180	Simplifying complex expressions involving mixed representational forms of complex numbers.	Interaction, level, guided reinvention, intertwinement.
6	45	Roots of complex numbers.	Activity, interaction, guided reinvention, intertwinement.

min, minutes; RME, realistic mathematics education.

The visual and enablement of GeoGebra were explored to create a complex number, $z_1 = 3 + 2i$. GeoGebra automatically creates the algebraic equivalence on the algebraic window. It feels like you have an assistant lecturer (see Figure 2, where the visual affordance of GeoGebra was explored to lessen the abstraction of a complex rectangular number).

Through Patsiomitou's (2012) theoretical dragging (purposeful transforming to acquire additional knowledge), the teacher explored GeoGebra's visual and enablement affordance and used the mouse to drag the complex number z_1. GeoGebra's visual and enablement affordance played the role of an amplifier and reorganiser (Sherman, 2014), mainly driven by interaction and guided principles of RME, enabling students to observe, explore, and engage with many examples of complex numbers in non-standard (imaginary unit j is implicit, see z_2; z_3; z_6 and where there is no real part, see z_4; z_8; z_9 in Figure 2) and standard (imaginary unit j is explicit) orientations in a coherent and continuous motion. The lecturer could have used a traditional whiteboard-and-marker approach as he did with CG. However, GeoGebra was used as an amplifier (which made the process more effective and efficient), lessening complex numbers' abstractions. GeoGebra helps students see abstract concepts (Antohe, 2009). This phase was intended to help more than 80% of students (Conner et al., 2007; Norlander & Norlander, 2012) associating a rectangular complex number with only the imaginary unit j.

Furthermore, unlike the conventional approach, where the formulas and rules (see Bird, 2017, pp. 255, 257, 261, 267, 268) are derived abstractly and socially shared with the passive students, the students engaged meaningfully in their modelling. More than an eighth (12.8%) of students in Norlander and Norlander's (2012) study indicated that they struggle to memorise complex number rules and formulas.

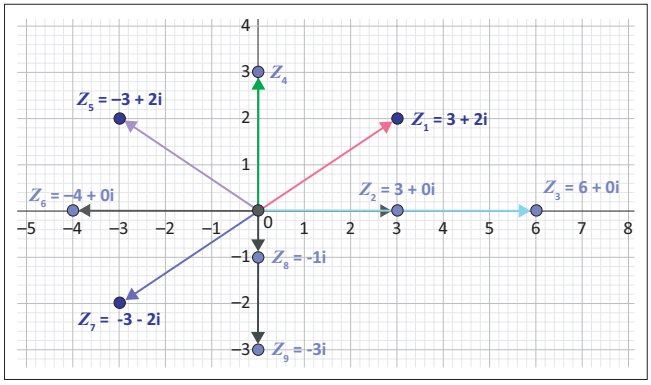


FIGURE 2: The summary of how the visual and enablement of GeoGebra was harnessed to show many examples of standard and non-standard complex numbers.

Hence, the visuals and enablement of GeoGebra were harnessed to model the sum of complex rectangular numbers.

The lecturer created two rectangular complex numbers, $z_1 = 3 + 2i$ and $z_2 = 1 + 3i$, in GeoGebra's graphic/geometrical window. He then typed the sum, $z_1 + z_2$ in the algebraic window. GeoGebra then gives the equation $z_3 = z_1 + z_2 = 4 + 5i$ in the same algebraic window and displays its equivalent $z_3 = 4 + 5i$ in the graphical window.

The lecturer explored GeoGebra's capabilities, slowly dragging the real part of z_2 in Figure 3 vertically while its imaginary part was fixed (see Figures 3a and 3b). He guided students through the interactions and guided reinvention principles of RME to link the visible geometric and algebraic changes of the real part of z_2 to the changes in the imaginary part of the dependent z_3.

Furthermore, the students engaged, self-discovered, and correctly linked the material numerical changes of the real part of z_3 in the algebraic and graphic windows when z_2 was dragged horizontally in the geometrical window (see Figures 3c and 3d). These processes and procedures are impossible with the traditional pen-and-paper approach.

The GeoGebra-facilitated activities played the role of Sherman's (2014) amplifier and reorganiser, providing students with ample opportunities for meaningful observations, explorations, engagements, physical interactions, self-discovery, and visualisations of the sum modelling of the rectangular form of complex numbers. Students could understand the algebraic and geometric representations of the sum of two complex rectangular numbers more than a mere sum of two binomials.

This process was abstracted through the level principle of RME to any number of complex rectangular numbers to enhance students' conceptual knowledge that given any two complex numbers, $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, the algebraic sum formula is $z_3 = (a_1 + a_2) + j(b_1 + b_2)$.

Figure 3 (a–d) summarises how GeoGebra's enablement and visualisation affordance was explored in modelling the linkage between the algebraic and geometric representations of the sum of complex numbers.

As indicated earlier, the study of Panaoura et al. (2006) showed that students fail to recognise that the complex number's rectangular and polar forms represent one entity. Smith et al. (2019) showed that 57.1% of students exhibited difficulties and misconceptions about the different complex number representations in a similar study. Another 12.8% of students

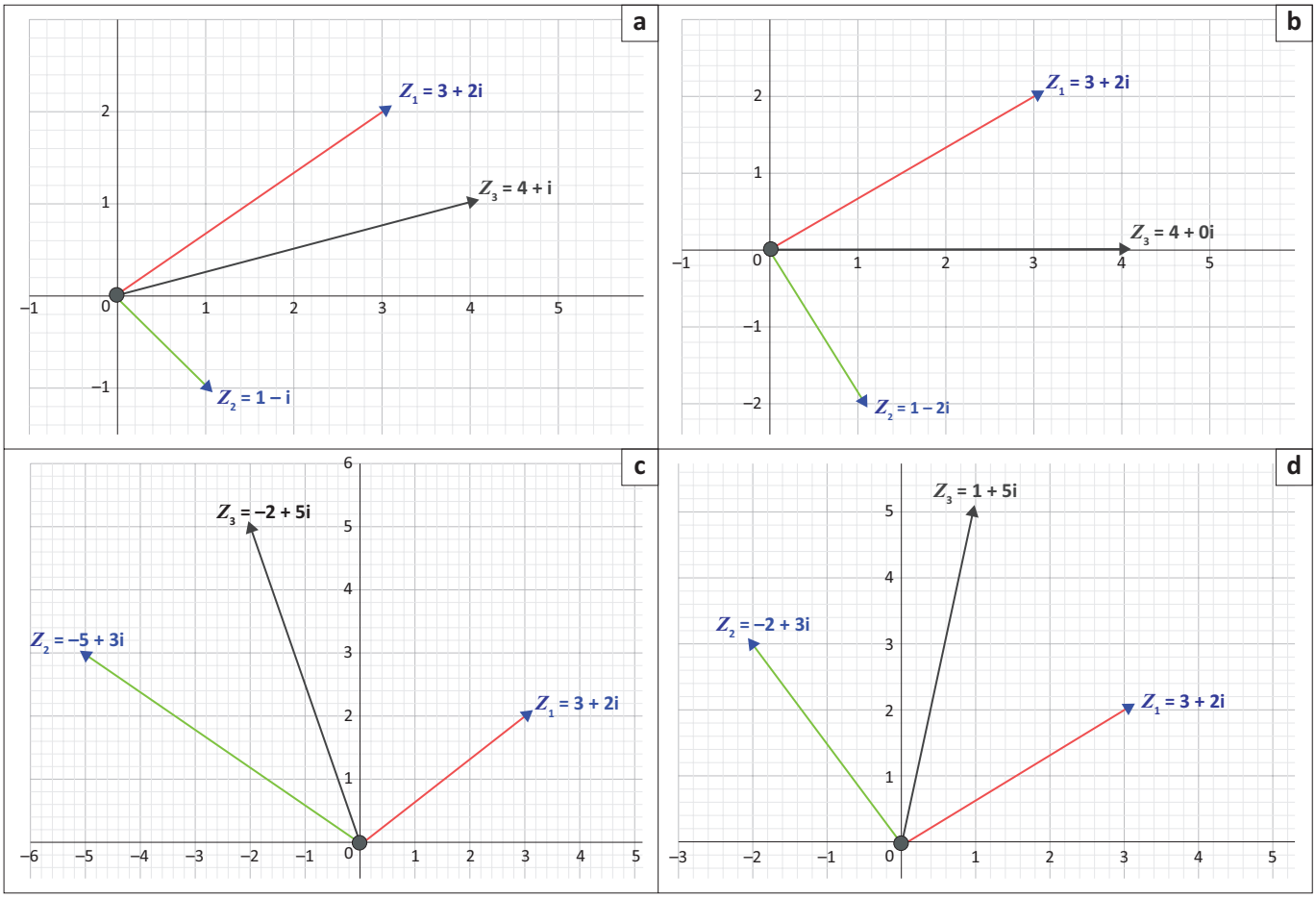


FIGURE 3: A summary of how the visual and enablement of GeoGebra was harnessed to show the geometrical representation of the sum of two rectangular complex numbers; (a and b) vertically; (c and d) horizontally.

mentioned difficulty managing polar or trigonometric representations (Smith et al., 2019).

These misconceptions and difficulties hinder the students in identifying, converting, and using the most appropriate representations where needed.

To fill this void, the lecturer used GeoGebra to create $z_1 = 2 + 2j$, measure its argument, 45°, (GeoGebra can also express the argument in radians) and modulus, 2.83. The rationale was for students to comprehend that z_1 can be expressed with two representations (i.e. $z_1 = 2 + 2i = [2.83; 45°]$). The lecturer-researcher intentionally manipulated and dragged $z_1 = 2 + 2i$ to create $3 + 3i$ (see Figure 4a). The rationale was for students to observe the modulus changing from 2.83 to 4.24 while the argument remained constant, creating $z = 3 + 3i = (4.24; 45°)$. The complex number $z_1 = 2 + 2i$ was further purposefully dragged into giving equations that equate the rectangular and polar representations, like $z_1 = 4 + 4i = (5.66; 45°)$; $z_1 = 5 + 5i = (7.07;45°)$. Moreover, $z_1 = 2 + 2i$ was dragged anticlockwise, giving equations like $z_1 = 2i = (2; 90°)$; $z_1 = -2-2i = (2,83;225°)$; $z_1 = -2 = (2;180°)$; $z_1 = -4i = (4;270°)$, et cetera. See Figure 4 (b–f) for a summary of how GeoGebra's enablement and visualisation capabilities were explored in connecting the algebraic and geometric representatives of the complex numbers.

These computations, explorations, demonstrations and procedures are only possible with technology like GeoGebra, and, according to Sherman (2014) and Pea (1987), allow students to focus primarily on observations, looking for patterns, identifying invariances or making and testing conjectures, giving students access to higher-level processes, extending their thinking, and thereby developing insight. As indicated earlier, the rationale was to assist the students in Panaoura et al.'s (2006) and Smith et al.'s (2019) studies to understand the different representations of a complex number not as two but as one merged coherent mathematics concept expressible in various representations. Therefore, feel free to convert between the different representations when necessary. The most dominant principles of RME were the guided reinvention and intertwinement (combined algebra, geometry, and trigonometry) principles of RME.

For a summary of the modelling of the quotient of complex numbers polar form, see Figure 7.

Figure 4a (Equation 4):

$$z_3 = \frac{z_1}{z_2} = \frac{(3.15;74.9°)}{(4;0°)} = \left(\frac{3.15}{4};74.9° - 0°\right) = (0.7875;74.9°)$$

[Eqn 4]

Figure 4b (Equation 5):

$$z_3 = \frac{z_1}{z_2} = \frac{(3.61;56.31°)}{(2.83;45°)} = \left(\frac{3.61}{2.83};56.31° - 45°\right) = (1.27;11.31°)$$

[Eqn 5]

The students self-discovered that, given any two polar complex numbers, $z_1 = (r_1 ; \theta_1)$ and $z_2 = (r_2 ; \theta_2)$, their quotient is (Equation 6):

$$z = \left(\frac{r_1}{r_2};\theta_1 - \theta_2\right)$$

[Eqn 6]

The product is $z = (r_1 \times r_2; \theta_1 + \theta_2)$ and the power is $(z_1)^n = ((r_1)^n; n\theta)$, where n is a positive integer.

It is important to note that the complex number's different parts (real and imaginary, moduli and arguments) were not confined to integers but included real numbers. Hence, the students conceptualised infinitely many examples.

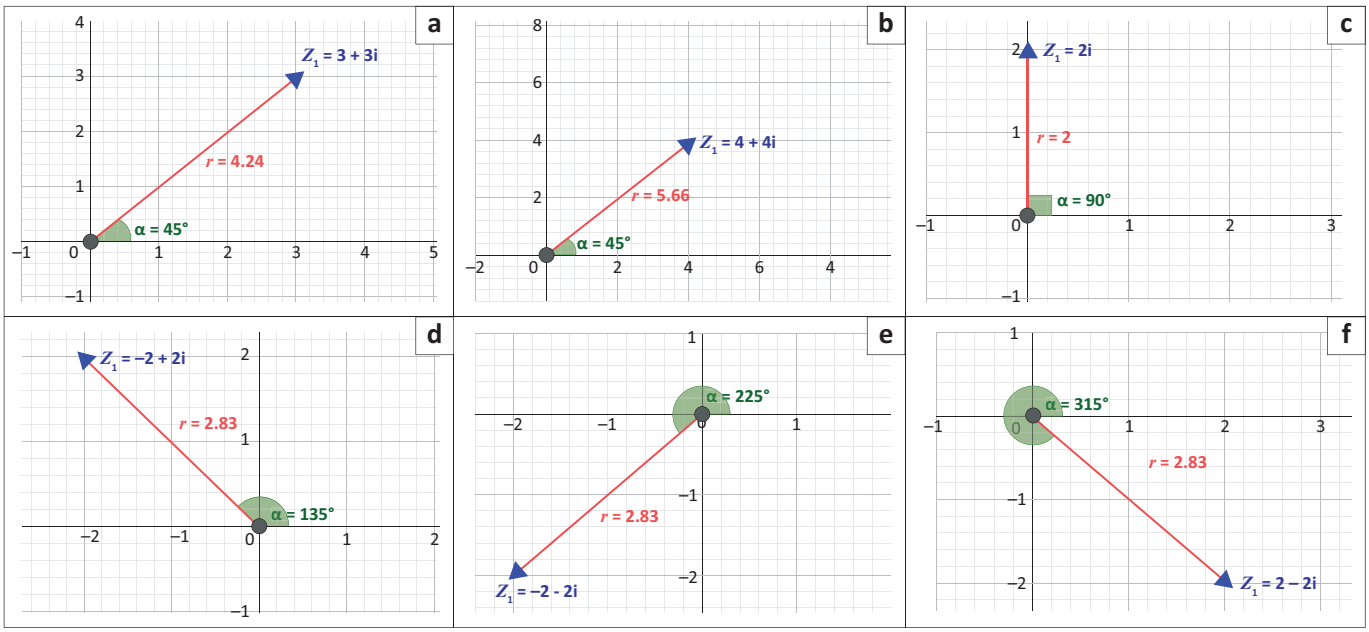
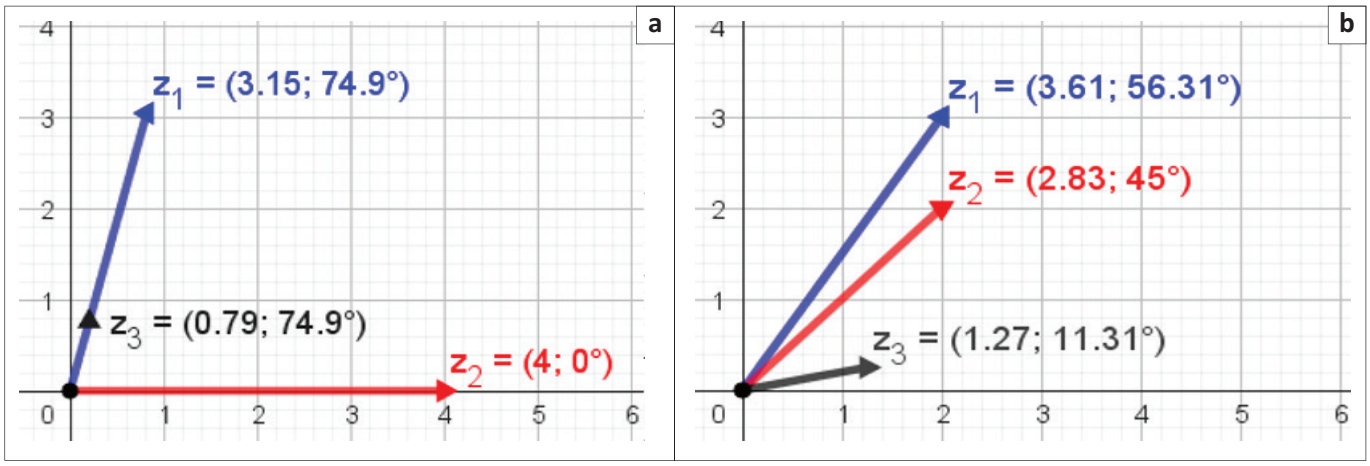


FIGURE 4: A summary of how (a) the visual and enablement of GeoGebra was harnessed to show (b-f) the linkage between a complex number's different representations.



Note: $z_3 = \frac{z_1}{z_2}$ for both Figure 5a and Figure 5b.

FIGURE 5: A summary of how the visual and enablement of GeoGebra was harnessed to model the quotient of polar complex numbers: (a) $\left[z_3 = \frac{z_1}{z_2} = \frac{(3.15;74.9°)}{(4;0°)} = \left(\frac{3.15}{4};74.9° - 0°\right) = (0.79;74.9°)\right]$ and (b) $\left[z_3 = \frac{z_1}{z_2} = \frac{(3.61;56.31°)}{(2.83;45°)} = (1.27;11.31°)\right]$.

Data collection

The first author (lecturer-researcher) marked students' pre-post test scripts following a predetermined rubric. Consistent accuracy (CA) marking was applied. If a student makes a mistake in a question and uses the incorrect answer or information correctly in the next step, they are marked positively.

The first author analysed the students' scripts for misconceptions and errors associated with complex numbers qualitatively, using Donaldson's (1963) and Orton's (1983) classification, and quantised or quantitised using the chi-square test.

The frequencies of errors in students' test scripts were used to establish whether GeoGebra as a modelling tool mitigates errors related to complex numbers compared to traditional complex numbers teaching. Two lecturers reviewed and moderated the test and students' test scripts to ensure validity and reliability. One lecturer was a regular lecturer of CG, and the other was a regular lecturer of another first-year first-semester engineering mathematics group. His group was not included in the study.

Data analysis

As indicated earlier, students' misconceptions and errors associated with complex numbers found on test scripts were analysed qualitatively and quantised or quantitised. Donaldson's (1963) and Orton's (1983) classification of errors, namely structural, arbitrary, and executive, were used as a lens to identify and classify the misconceptions and errors.

A chi-square test of independence was performed and tested at a significance level of $p = 0.05$. The assumption was that there would not be any significant difference between the frequency of EG and CG students' misconceptions and errors after their respective interventions. Therefore, this test aimed to determine if EG's GeoGebra-facilitated intervention mitigated misconceptions and errors associated with complex numbers better than CG's conventional intervention. Furthermore, the test will determine whether any difference between EG's and CG's misconceptions and errors is due to chance.

Results

Both groups benefitted from their respective interventions. The frequency of student total errors decreased from the pre test to the post test.

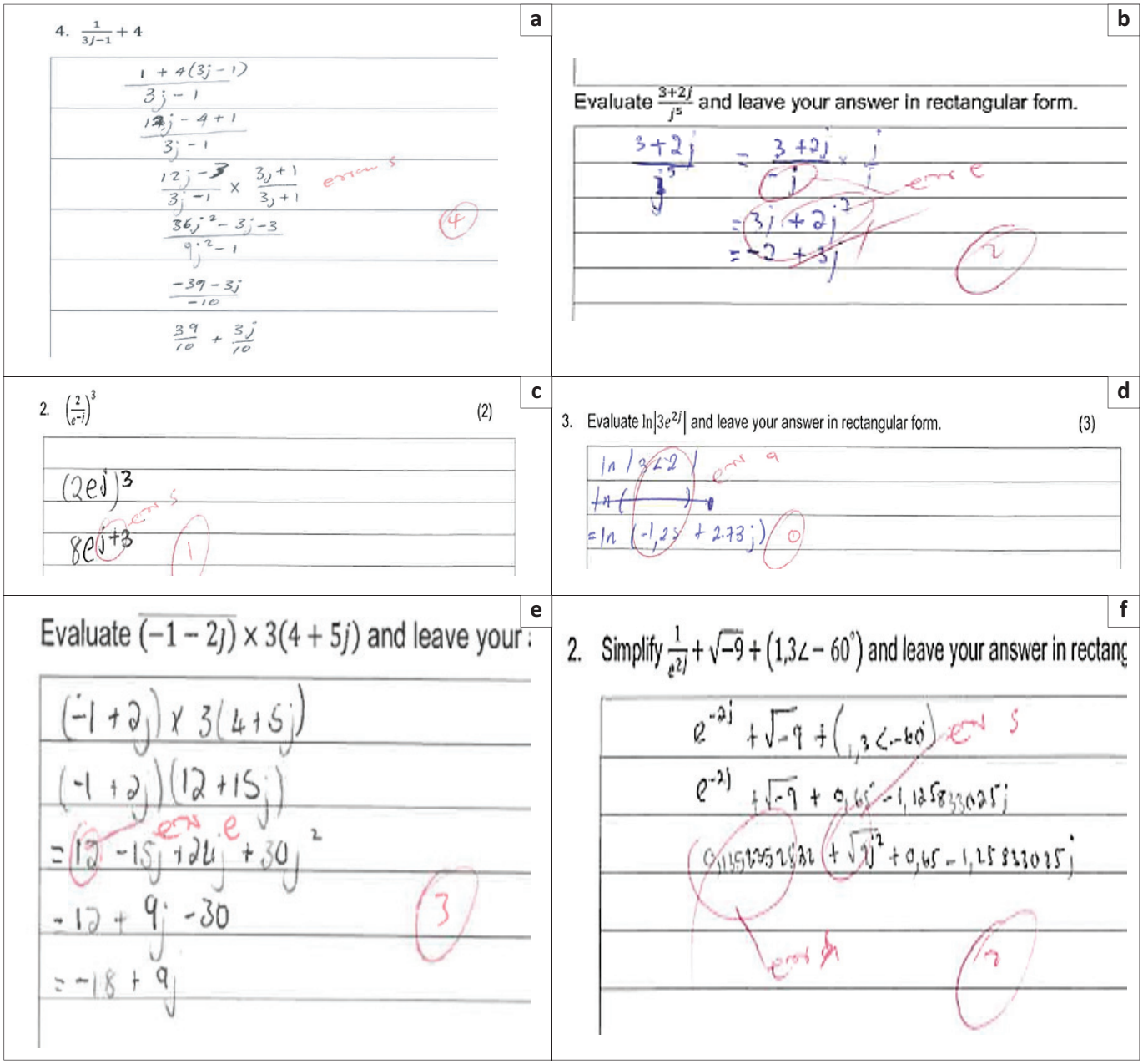
Figure 6 shows EG and CG vignette sample scripts displaying post-intervention misconceptions and errors.

Table 2 summarises the CG and EG students' script errors shown in Figure 6, post-intervention, using Orton and Donaldson as a lens.

Item Figure 6a was on the conceptual knowledge of the imaginary unit j, the complex conjugate and the subsequent algebraic algorithmic for the simplifications to $a + bj$.

The EG student vignette shows that the student incorrectly gave the complex conjugate of $3j-1$ as $3j+1$, suggesting a misconception that a complex conjugate of a complex rectangular number is the same complex rectangular number with the middle sign changed. This error seems to be systematic. The student failed to harness GeoGebra's visual affordance because the complex conjugate and the complex number reflect each other on the real axis. However, it should be noted that the student, unlike those in many different studies (e.g., Ahmad & Shahrill, 2014; Norlander & Norlander, 2012), did not have difficulties handling the imaginary unit, j. The student's algebraic skill seems solid.

Figure 6c was based on conceptual knowledge of exponential manipulation. The EG student vignette shows a fragmented knowledge of exponential functions. The student correctly converted the exponential representation $\frac{2}{e^{-j}}$ into $2e^{j}$ and found $(2)^3 = 8$. However, the student wrote $(e^{j})^3$ incorrectly as



Note: Figure 6a, c and f show students' structural errors. Figure 6d is an overlap of structural and arbitrary errors. Figure 6b and e is executive error.

FIGURE 6: Vignettes of (a, c & e) EG and (b, d & f) CG students showing post-intervention errors.

TABLE 2: Classification of errors found on sampled students' post test scripts.

Group	Item	Description	Type of errors		
			Structural	Executive	Arbitrary
EG	Figure 6a	Simplify to $a + bj$	√	-	-
	Figure 6c	Simplify to $re^{j\theta}$	√	-	-
	Figure 6e	Simplify an expression that goes beyond algebraic knowledge.	-	√	-
CG	Figure 6b	Simplify to $a + bj$.	-	√	-
	Figure 6d	Choice of appropriate representation and simplify to $a + bj$.	√	-	√
	Figure 6f	Simplifying an expression that contains multiple complex number representations.	√	-	-

Note: Please see the full reference list of the article, Seloane, P.M., Ramaila, S., & Ndlovu, M. (2025). Harnessing GeoGebra as a modelling tool to mitigate undergraduate engineering mathematics students' misconceptions and errors associated with complex numbers. *Pythagoras, 46*(1), a825. https://doi.org/10.4102/pythagoras.v46i1.825, for more information.

EG, experimental group; CG, control group.

e^{j+3} instead of e^{3j}. The student seems to have a misconception about the two exponential laws: $(a^n \cdot b^m)^x = a^{nx} \cdot b^{xm}$ and $a^n \cdot a^m = a^{n+m}$. Carraher and Schliemann (2007) also observed that many students have difficulty remembering and applying the exponential laws correctly. Hence, the error is systematic. Please see the discussion for more CG and EG students' systematic errors on this item, Figure 6c.

Item Figure 6e was about the conceptualisation of complex conjugate and its notation, multiplication of a complex number

by a real number, handling of the imaginary unit, j, and general complex number algebraic manipulation. The EG vignette suggests that GeoGebra-facilitated intervention helped the student to acquire the conceptual and procedural knowledge required for the task. GeoGebra-enriched activities helped students to conceptualise and visualise a complex rectangular number, $z = a + bj$, including the role and meaning of j in contrast to Ahmad and Shahrill's (2014) study. While the student's procedural knowledge of the rectangular form of a complex number seemed solid, an error was made. The student wrote $(-1 + 2j)(12 + 15j) = 12-15j + 24j + 30j^2$. The error (12) does not seem structural because the other three terms, $-15j + 24j + 30j^2$, in the following expression, $12-15j + 24j + 30j^2$, including all the subsequent expressions, are all correct. This suggests that the student understood all the relevant principles and only failed to manipulate them, resulting in executive error. Please see the discussion for more CG and EG students' systematic errors on this item (Figure 6e).

The item Figure 6b is based on the conceptualisation of the imaginary unit, j, conjugates, and the manipulation of fractions containing the powers of j. The CG student displayed a solid arithmetic operation of rectangular complex numbers, including the role of the imaginary unit j. The student used the correct conjugate of j, correctly multiplied (3+ 2j) by j, and substituted j^2 with −1. However, the student wrote $j^5 = -j$ instead of $j^5 = j$. The error seems more of an executive error than a systematic one.

The item Figure 6d was on students' choice of the appropriate complex number representation to perform a particular task. The CG vignette suggests that the student needs to know more about what needs to be done. Hence, the student got 0% (0 out of 3) for this test item. The student was, first, supposed to continue with the exponential representation, use the product law for logarithmic functions and expand as $\ln|3e^{2j}| = ln(3) + ln(e^{2j})$. Secondly, they needed to simplify the second term, $ln(e^{2j})$, using the power law for logarithmic functions to get $ln|3e^{2j}| = ln\ (3) + 2jln(e)$. And, lastly, they had to apply the properties of logarithmic functions to obtain the correct answer as $ln|3e^{2j}| = ln(3) + 2j$. Although the student correctly converted $3e^{2j}$ to 3 < 2, and 3 < 2 to −1,248 + 2,7278j this wasn't very sensible in the task context. The error seems to be an overlap of systematic and arbitrary. However, it appears to be skewed more toward systematic.

Orton (1983) also noticed some error overlaps. The student was supposed to know the appropriate representation for the given task and when and how to convert. Panaoura et al. (2006) and Smith et al. (2019) also observed the same in their studies. Please see the discussion for more on CG and EG students' systematic and arbitrary errors on this item (Figure 6d).

The item Figure 6f is a complex number expression containing multiple complex number representations (i.e., exponential, rectangular and polar). Moreover, the argument in polar is measured in degrees. Therefore, it needs several areas of expertise, namely the choice of the correct representation and how and when to convert the others into the appropriate representation. Although the student, unlike in Ramaila and Seloane's (2018) and Panaoura et al.'s (2006) studies, was aware that the most suitable representation to simplify this expression containing multiple representations is rectangular and correctly converted the polar into rectangular; however, in their quest to convert the exponential representation and finally simplify to the correct answer, they committed several systematic errors. The student substituted −1 with j^2 and wrote $\sqrt{-9} = \sqrt{9j^2}$, ignored j in e^{-2j} and wrote $e^{-2j} = 0.1353352$. Again, the student, like those in other studies (e.g., Ahmad & Shahrill, 2014; Hui & Lam, 2013; Smith et al., 2019), displayed difficulties and misconceptions about the role of j. Please see the discussion for more CG and EG students' systematic errors on this question item (Figure 6f).

Table 3 summarises the type and frequencies of errors found on students' scripts for pre-post test (1) on complex rectangular form using Orton and Donaldson as a lens.

Significance testing for test 1 systematic errors yields a chi-square statistic of 1.7561, whose p-value is 0.185109 and not significant at $p < 0.05$.

Significance testing for test 1 arbitrary errors yields a chi-square statistic of 0.7119 and a p-value of 0.398815, which is not significant at $p < 0.05$.

Significance testing of test 1 executive errors yielded a chi-square statistic of 3.0731 and a p-value of 0.079596, which was insignificant at $p < 0.05$.

However, significance testing for all total test 1 errors yielded a chi-square statistic of 5.6807 and a p-value of 0.017153, which was significant at $p < 0.05$. This means the ratios 128:47 and 125:78 are statistically significantly different. Hence, the observed differences in students' total test 1 errors did not occur by chance.

A similar table was done for pre-post test (2), and the following is a summary of the type and frequencies of errors found on students' test scripts on complex numbers:

- Significance testing for systematic errors yields a chi-square statistic of 1.1227, whose p-value is 0.289345 and not significant at $p < 0.05$.

TABLE 3: Frequencies of errors in Test 1.

Type of errors	EG Pre-test (26 students)	CG Pre-test (27 students)	EG Post-test (26 students)	CG Post-test (27 students)
No. of systematic errors	52	56	16	28
No. of arbitrary errors	34	27	8	10
No. of executive errors	37	32	19	28
No. of no attempts	5	10	4	12
Total number of errors	128	125	47	78
Average	4.92	4.63	1.81	2.9

EG, experimental group; CG, control group; No., number.

- Significance testing for test 2 arbitrary errors yields a chi-square statistic of 2.6389 and a *p*-value of 0.104278, which is insignificant at $p < 0.05$.
- Significance testing of test 2 executive errors yielded a chi-square statistic of 0.1043 and a *p*-value of 0.746769, which was insignificant at $p < 0.05$.
- Finally, significance testing for the total test 2 errors yielded a chi-square statistic of 3.0731 and a *p*-value of 0.079596, which was not significant at $p < 0.05$. This means the ratios 234:88 and 259:130 are not statistically significantly different. Hence, the observed differences in students' total test 2 errors occurred by chance.

Discussion

Table 3 shows that for test 1, EG students made 52 (average of 2) systematic errors compared to 56 (average of 2.07) by the CG before the instructional intervention. This, however, changed to 16 (average of 0.0062) and 28 (average of 1.04) for EG and CG. Using GeoGebra reduced the EG's systematic errors by 69% (from 52 to 16) compared to the CG's conventional approach of 50% (from 56 to 28). Using GeoGebra further reduced the EG's arbitrary errors by 76% compared to the CG's traditional teaching and learning of 63%. For the EG and the CG, executive errors decreased by 49% and only 13%. The number of 'no attempts' by the EG was reduced by 20%, while the CG recorded an increase of 20%.

The GeoGebra-enriched instruction helped to reduce the total number of test 1 errors by 63.3% (from 128 to 47), while conventional instruction managed only 37.6% (from 125 to 78).

However, as indicated earlier, significance testing for the total test 1 errors yielded a chi-square statistic of 5.6807 and a *p*-value of 0.017153, which was significant at $p < 0.05$. Therefore, for test 1, GeoGebra-facilitated intervention mitigated undergraduate engineering mathematics students' errors associated with the rectangular complex number better than conventional teaching and learning. This inference is consistent with several studies which demonstrated that GeoGebra positively impacts the teaching and learning of many mathematical topics, (e.g., Amam et al., 2017; Bulut et al., 2016; Karakok et al., 2014; Kin, 2018; Monika & Horacio, 2018; Ogbonnaya & Mushipe, 2020; Pjanic & Liden, 2015).

For test 2, GeoGebra intervention reduced systematic errors remarkably by 68% (from 125 to 40) compared to traditional teaching and learning's 59% (from 133 to 55). The number of arbitrary errors for the EG significantly decreased by 70% compared to 49% for the CG. Executive errors increased by 24% for the EG and and 8% for the CG. The number of 'no attempts' decreased by 75% and 60% for the EG and the CG.

As indicated earlier, the significance testing for the total test 2 errors yielded a chi-square statistic of 3.0731 and a *p*-value of 0.079596, which was insignificant at $p < 0.05$. Thus, the observed error differences occurred by chance.

However, significance testing for the combined (test 1 and test 2) errors yielded a chi-square statistic of 7.9584 and a *p*-value of 0.004787, which was significant at $p < 0.05$. This means the ratios 362:135 and 384:208 are statistically significantly different.

Thus, the observed error differences in the combined tests (test 1 and test 2) did not occur by chance. What is also noticeable is that:

- Seven students (EG and CG) (13.2%) struggled with *j*. One student from each group left $\sqrt{-9}$ as is, two EG and one CG student wrote $\sqrt{-9} = 9j$, one EG student wrote $\sqrt{-9} = -9j$ and one CG student wrote $\sqrt{-9} = \sqrt{9j^2}$. This raises questions about their conceptual knowledge of *j* and is line with Ahmad and Shahrill's (2014) observations that students' overreliance on a calculator makes them struggle with the square root of a negative number. Strangely, all these seven students correctly answered the item $\overline{(-1-2j)}\cdot 3(4+5j)$, which involved the manipulation of *j*.
- Many students (EG and CG) displayed difficulties and struggled with items involving complex number domain concepts like logarithmic, exponential, and trigonometric functions. The item, $ln|3e^{2j}|$, recorded more misconceptions or systematic errors than any other test item. A total of 8 students (2 EG and 6 CG) gave no response in the post test. This contributed 44.4% of the 14 'no attempts' in the test 2 post test. Only 31% (8 out of 26) and 37% (10 out of 27) from the EG and the CG answered this item correctly. Of the remaining students, 62% (16 out of 26) of the EG and 41% (11 out of 27) of the CG had either a systematic or arbitrary error. Some notable systematic errors included:

  $ln|3e^{2j}| = 2jln|3e|$; $ln|3e^{2j}| = ln3 + e^{2j}$;

  $ln|3e^{2j}| = 2jlog_e 3e$, $ln|3e^{2j}| = \frac{1}{3e^{2j}} = 3e^{-2j} = 3e-2j$.

  This is concerning as, according to DBE (2011), logarithmic and exponential functions are covered in Grades 10–12. However, Machaba (2016) and Roselizawati et al. (2014) remind us that misconceptions can arise when students try or fail to construct and organise new knowledge using prior knowledge. Smith et al. (1993) further argue that students can firmly hold misconceptions and be apathetic toward any instructions to correct them.
- Some students ignored the degree unit in the item $(2 < 30°)^3$ during their workings, while others only brought it back on their final answer. Some did not convert the degree unit in the denominator to radians in the item,

  $$\frac{(1\angle 1,2)^3 (2\angle 4,23)^2}{(3\angle -25,78^\circ)^3}$$

  They ignored the degree unit and continued working without it. Some inserted the degree unit only at the end.

This indicates the misconception between the degrees and radians measures of the argument, θ. Some showed systematic errors, as follows:

$$\left(\frac{2}{e^j}\right)^3 = \left(\frac{2}{e-j}\right)^3 = (2e+j)^3, \left(\frac{2}{e^j}\right)^3 = 8e^{j+3}$$

suggesting a shaky conceptual and procedural exponential knowledge. This, again, is concerning as, according to DBE (2011), exponential functions are covered in Grades 10–12.

Six students, three from each group, got stuck and wrote it as follows:

$$\frac{1}{e^{2j}} = \frac{1}{1<2}$$

This wrong choice of the complex number's form, also observed by Panaoura et al. (2006) and Smith et al. (2019), impacted their quest to convert from exponential to rectangular form.

Conclusion and recommendations

The study's findings suggest that GeoGebra-facilitated intervention effectively mitigates undergraduate engineering mathematics students' total errors associated with complex numbers. Due to the limited number of participants, the study could be extended, and the efficacy of GeoGebra could be investigated to include many participants and the remaining three university groups. More complex numbers' subtopics could also be included, and GeoGebra's effect could be investigated for more than two weeks in line with previous results (e.g. Chan & Leung, 2014; Juandi et al., 2021); the positive treatment results tend to fade as the duration is extended. The new treatment can motivate students (Juandi et al., 2021).

The study could also be repeated in a computer lab where all participants can access GeoGebra. However, GeoGebra-facilitated intervention was ineffective in mitigating students' executive errors. This finding aligns with the 20% (4 out of 20 studies) of Uwurukundo et al.'s (2020) study that found GeoGebra ineffective.

However, for this study, further empirical investigations could be done on why GeoGebra was ineffective in mitigating students' executive errors, mainly because the frequency of executive errors in test 2 increased from pre test to post test for both groups.

Students demonstrated difficulties when manipulating logarithmic, exponential, and trigonometric functions. This deficiency negatively affected EG and CG students' learning of complex numbers. Therefore, lecturers are encouraged to help their students revise these domain topics before and even while teaching complex numbers.

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Authors' contributions

M.P.S. was responsible for conceptualisation, methodology, formal analysis, investigation and writing the original draft. S.R. and M.D. were responsible for writing, review and editing, and supervision.

Ethical considerations

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Data availability

Data sharing is not applicable to this article as no new data were created or analysed in this study.

Disclaimer

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References

Ahmad, A.W., & Shahrill, M. (2014). Improving post-secondary students' algebraic skills in the learning of complex numbers. *International Journal of Science and Research, 3*(8), 273–279.

Amam, A., Fatimah, A.T., Hartono, W., & Effendi, A. (2017). Mathematical understanding of the underprivileged students through GeoGebra. *Journal of Physics Conference Series, 895*, 012007. https://doi.org/10.1088/1742-6596/895/1/012007

Anevska, K., Gogovska, V., & Malcheski, R. (2014). The role of complex numbers in interdisciplinary integration in mathematics teaching. *Procedia – Social and Behavioral Sciences, 191*, 2573–2577. https://doi.org/10.1016/j.sbspro.2015.04.553

Antohe, V. (2009). Limits of educational soft "GeoGebra" in a critical constructive review. Anale. *Seria Informatică, VII*, 47–54.

Barlow, A.T., Watson, L.A., Tessema, A.A., Lischka, A.E., & Strayer, J.F. (2018). Carefully select and leverage student errors for whole – Class discussion to benefit the learning of all. *Teaching Children Mathematics, 24*(6), 384–391. https://doi.org/10.5951/teacchilmath.24.6.0384

Bird, J. (2017). *Higher engineering mathematics* (8th ed.). Taylor & Francis Group.

Bulut, M., Akcakin, H.U., Kaya, G., & Akcakin, V. (2016). The effects of GeoGebra on third grade primary students' academic achievement in fractions. *International Electronic Journal of Mathematics Education, 11*(2), 347–355. https://doi.org/10.29333/iejme/338

Carraher, D., & Schliemann, A. (2007). Early algebra and algebraic reasoning. In F.K. Lester (Ed.), *Second handbook of research of mathematics teaching and learning* (pp. 669–706). Information Age.

Chan, K.K., & Leung, S.W. (2014). Dynamic geometry software improves mathematical achievement: Systematic review and meta-analysis. *Journal of Educational Computing Research, 51*(3), 311–325. https://doi.org/10.2190/EC.51.3.c

Colin, F. (2023). Methodological pragmatism in educational research: From qualitative-quantitative to exploratory-confirmatory distinctions. *International Journal of Research & Method in Education, 47*(1), 4–19. https://doi.org/10.1080/1743727X.2023.2210063

Conner, E., Rasmussen, C., Zandieh, M., & Smith, M. (2007). Student understanding of complex numbers. In *Electronic proceedings for the 10th special interest group of the mathematical association of America on research in undergraduate mathematics education*. Retrieved from http://sigmaa.maa.org/rume/crume2007/papers/conner-rasmussen-zandieh-smith.pdf

Creswell, J.W. (2018). *Research design: Qualitative, quantitative, and mixed methods approach* (5th ed.). Sage.

Department of Basic Education (DBE). (2011). *National curriculum statement: (Further education and training phase grades 10–12)*. Department of Education.

Diković, L. (2009). Applications GeoGebra into teaching some topics of mathematics at the college level. *Computer Science and Information Systems, 6*(2), 191–203. https://doi.org/10.2298/CSIS0902191D

Donaldson, M. (1963). *A study of children's thinking*. Tavistock Publications.

Drijvers, P. (2012). Digital technology in mathematics education: Why it works (or doesn't). In *Selected regular lectures from the 12th international congress on mathematical education, 08–15 July 2012* (pp. 135–151). Springer International Publishing.

Esguerra-Prieto, B., González-Garzón, N., Acosta-López, A. (2018). Mathematical software tools for teaching complex numbers. *Revista Facultad de Ingenieria, 27*(48), 79–90. https://doi.org/10.19053/01211129.v27.n48.2018.8403

Hohenwarter, M., & Fuchs, K. (2004). Combination of dynamic geometry, algebra and calculus in the software system GeoGebra. *Zdm,* 128–133. Retrieved from https://api.semanticscholar.org/CorpusID:15254420

Hohenwarter, M., Hohenarter, J., Kreis, Y., & Lavicza, Z. (2008). Teaching and calculus with free mathematics software GeoGebra. In *11th international congress on mathematical education, Monterrey, Mexico, 08 July 2008*. International Congress on Mathematical Education (ICME). Retrieved from https://www.researchgate.net/publication/228869636_Teaching_and_Learning_Calculus_with_Free_Dynamic_Mathematics_Software_GeoGebra

Hui, T.S., & Lam, T.T. (2013). On the teaching of the representation of complex numbers in the argand diagram. *Learning Science and Mathematics, 8*, 75–86.

Ibrahim, K., & Ilyas, Y. (2016). Teaching a concept with GeoGebra: Periodicity of trigonometric functions*. *Educational Research and Reviews, 11*(8), 573–581. https://doi.org/10.5897/ERR2016.2701

Juandi, D., Kusumah, Y.S., Tamur, M., Krisna S., Perbowo, M., & Wijaya, T.T. (2021). A meta-analysis of GeoGebra software decade of assisted mathematics learning: What to learn and where to go?. *Heliyon, 7*(5), e06953. https://doi.org/10.1016/j.heliyon.2021.e06953

Karakok, G., Soto-Johnson, H., & Dyben, S.A. (2014). Secondary teachers' conception of various forms of complex numbers. *Journal of Mathematics Teacher Education, 18*, 327–351. https://doi.org/10.1007/s10857-014-9288-1

Kaushik, V., & Walsh, C.A. (2019). *Pragmatism as a research paradigm and Its Implications for social work research*. Faculty of Social Work, University of Calgary.

Keeley, P. (2012). *Misunderstanding misconceptions*. Guest Editorial. New Science Teacher Academy.

Keeley, P., Eberle, F., & Tugel, J. (2007). *Uncovering student ideas in science: 25 more formative assessment probes*. NSTA Press.

Kin Eng, C., & Fui Fong, J. (2020). Knowing and grasping of two university students: The case of complex numbers. *Mathematics Enthusiast, 17*(1), 11. https://doi.org/10.54870/1551-3440.1487

Kin, K.P. (2018). Learning fraction comparison by using a dynamic mathematics software-GeoGebra. *International Journal of Mathematical Education in Science and Technology, 59*(3), 469–479. https://doi.org/10.1080/0020739X.2017.1404649

Lederman, N.G., & Niess, M.L. (2000). Technology for technology's sake or for the improvement of teaching and learning?. *School of Science and Mathematics, 100*(7), 345–348. https://doi.org/10.1111/j.1949-8594.2000.tb18175.x

Luneta, K. (2008). Error discourse in fundamental physics and mathematics: Perspectives of students' misconceptions. In *ICET 2008 International Council on Education for Teaching (ICET) 2008 international yearbook* (pp 386–400). National–Louis University Press.

Luneta, K. (2015). Understanding students' misconceptions: An analysis of final Grade 12 examination questions in geometry. *Pythagoras, 36*(1), 1–11. https://doi.org/10.4102/pythagoras.v36i1.261

Machaba, F.M. (2016). The concepts of area and perimeter: Insights and misconceptions of Grade 10 learners. *Pythagoras, 37*(1), a304. https://doi.org/10.4102/pythagoras.v37i1.304

Matlala, S., & Luneta, K. (2018). Students' misconceptions and errors are displayed when simplifying exponential expressions. In R. Govender & K. Junqueira (Eds.), *Proceedings of the 24th Annual National Conference of AMESA Culture meets culture, mathematics in and around us, 25–29 June 2018* (pp. 307–319). University of the Free State.

McNamara, J., & Shaughnessy, M.M. (2011). *Students' errors: What can they tell us about what students do understand*. Retrieved from http://www.mathsolutions.com/documents/studenterrors.

Monika, D., & Horacio, S. (2018). ICT integration in mathematics initial teacher training and its impact on visualization: The case of GeoGebra. *International Journal of Mathematical Education in Science and Technology, 49*(1), 66–84. https://doi.org/10.1080/0020739X.2017.1341060

Norlander, M. & Norlander, E. (2012). On the concept image of complex number. *International Journal of Mathematics Education in Science and Technology*, *43*(5), 627–641. https://doi.org/10.1080/0020739X.2011.633629

Ocal, M.F. (2017). The effect of Geogebra on students' conceptual and procedural knowledge: The case of applications of derivative. *Higher Education Studies, 7*(2), 67–78. https://doi.org/10.5539/hes.v7n2p67

Ogbonnaya, U.I. & Mushipe, M. (2020). The efficacy of GeoGebra-assisted instruction on students' drawing and interpretations of linear functions. *International Journal of Learning, Teaching and Educational Research, 19*(9), 1–14. https://doi.org/10.26803/ijlter.19.9.1

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics, 14*, 1–18. https://doi.org/10.1007/BF00704699

Panaoura, A., Elia, I., Gagatsis, A., & Giatilis, G.P. (2006). Geometric and algebraic approaches in the concept of complex numbers. *International Journal of Mathematical Education in Science and Technology, 37*(6), 681–706. https://doi.org/10.1080/00207390600712281

Patsiomitou, S. (2011) Theoretical dragging: A non-linguistic warrant leading to dynamic propositions. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, Ankara* (Vol. 3, pp. 361–368). Professional Military Education (PME).

Pea, R.D. (1987). Cognitive technologies in mathematics education. In A.H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 89–122). Erlbaum.

Pfeiffer, C. (2017). *A study of the development of mathematical knowledge in a GeoGebra-focused learning environment*. Doctoral thesis. Stellenbosch University.

Phan-Yamada, T., & Man, S.W. (2018). Teaching statistics with GeoGebra. *North American GeoGebra Journal, 7*(1), 14–24.

Pjanic, K., & Liden, E. (2015). One usage of GeoGebra in enhancing pre-service mathematics teachers' content knowledge. *Turkish Journal of Computer and Mathematics Education, 6*(1), 18–30. https://doi.org/10.16949/turcomat.78085

Ramaila, S., & Seloane, P. (2018). South African undergraduate engineering students' understanding of complex numbers. In *Proceedings of the 11th annual international conference of education, research and innovation*.

Riccomini, P.J. (2005). Identification and remediation of systematic error patterns in subtraction. *Learning Disability Quarterly, 28*(3), 233–242. https://doi.org/10.2307/1593661

Roselizawati, H., Sarwadi, H., & Shahrill, M. (2014). Understanding students' mathematical errors and misconceptions: The case of year 11 repeating students. *Mathematics Education Trends and Research, 2014*, 1–10. https://doi.org/10.5899/2014/metr-00051

Rushton, S.J. (2018). Teaching and learning mathematics through error analysis. *Journal of Cheminformatics, 3*(4), 1–12. https://doi.org/10.1186/s40928-018-0009-y

Sandelowski, M., & Voils, C.I., & Knafl, G. (2009). On quantitizing. *Journal of Mixed Methods Research, 3*(3), 208–222. https://doi.org/10.1177/1558689809334210

Shadaan, P., & Eu, L.K. (2013). Effectiveness of using GeoGebra on students' understanding in learning circles. *Malaysian Online Journal of Educational Technology, 1*(4), 1–11.

Sherman, M. (2014). The role of technology in extending students' mathematical thinking: Extending the metaphors of amplifier and reorganizer. *Contemporary Issues in Technology and teacher Education, 14*(3), 220–246.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145. https://doi.org/10.2307/749205

Smith, E.M., Zwolak, J.P., & Manogue, C.A. (2016). *Student difficulties with complex numbers*. American Association of Physics Teachers under a Creative Commons Attribution 3.0 license.

Smith, E.M., Zwolak, J.P., & Manogue, C.A. (2019). Isolating approaches: How middle-division physics students coordinate forms and representations in complex algebra. *Physical Review Physics Education Research, 15*, 010138. https://doi.org/10.1103/PhysRevPhysEducRes.15.010138

Smith, J.P., DiSessa, A.A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *Journal of the Learning Sciences, 3*(2), 115–163. https://doi.org/10.1207/s15327809jls0302_1

Supriadi, N., Kusumah, Y.S., Sabandar, J., & Afgani, J.D. (2014). Developing high-order mathematical thinking competency on high school students' through GeoGebra-assisted blended learning. *Mathematical theory and Modeling, 4*(6), 57–65.

Tay, M.K., & Mensah-Wonkyi, T. (2018). Effect of using GeoGebra on senior high school students' performance in circle theorems. *African Journal of Educational Studies in Mathematics and Sciences, 14*, 1–17.

Uwurukundo, M.S., Maniraho, F.S., & Tusiime, M. (2020). GeoGebra integration and effectiveness in the teaching and learning of mathematics in secondary schools: A review of literature. *African Journal of Educational Studies in Mathematics and Sciences, 16*(1), 1–13. https://doi.org/10.4314/ajesms.v16i1.1

Veloo, A., Krishnasamy, H.N., & Abdullah, W.S.W. (2015). Types of student errors in mathematical symbols, graphs and problem-solving. *Asian Social Science, 11*(15), 324–334. https://doi.org/10.5539/ass.v11n15p324

Veith, J.M., & Bitzenbauer, P. (2021). Two challenging concepts in mathematics education: Subject-specific thoughts on the complex unit and angles. *European Journal of Science and Mathematics Education, 9*(4), 244–251. https://doi.org/10.30935/scimath/11251

Zulnaidi, H., & Zamri, S.N.A.S. (2016). The effectiveness of the GeoGebra software: The intermediary role of procedural knowledge on students' conceptual knowledge and their achievement in mathematics. *EURASIA Journal of Mathematics Science and Technology Education, 13*(6), 2155–2180. https://doi.org/10.12973/eurasia.2017.01219a