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Analysis of preservice teachers' understanding of solving trigonometric equations: A perspective through actions, processes, objects, and schemas theory

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This study analysed first-year preservice teachers' understanding of trigonometric equations at a South African university in the Eastern Cape province. We employed the Action-Process-Object-Schema (APOS) framework to analyse the mental constructions made by preservice teachers in solving trigonometric equations. A qualitative case study design was employed to analyse test scripts from 223 preservice teachers, complemented by follow-up interviews with eight of these participants. Findings show that the success rate in the two analysed items was low. Students who had not developed specific mental structures could not solve the given problems. Only 15.5% of the participants reached the Object level, while 76% remained at the Action or Process stages. Conversely, 8.5% of the participants were at the pre-Action stage, having not shown evidence of action mental structures conjectured in the genetic decomposition. Challenges encountered include difficulties with algebraic manipulations, reference angles, angle relationships across quadrants, and conversions between degrees and radians. The analysis further revealed a lack of understanding of the periodic nature of trigonometric functions and the general solution derivation.

Contribution: These findings reflect global trends in mathematical struggles across various educational levels, particularly in solving trigonometric equations. The study highlights the importance of assessing preservice teachers' mathematical knowledge both at the entry and exit points of their training programmes. Such dual assessments could improve their content mastery and teaching effectiveness. This suggests that adjusting educational strategies to address these identified gaps could foster significant growth.

Keywords: APOS theory; preservice teachers; trigonometric equations; radian measure; algebraic method; genetic decomposition; mental structures.

Introduction

Trigonometry is a mathematical discipline that explores the connections between the sides and angles of triangles. It utilises angle measures, whether in radians or degrees, to determine point coordinates on circles or to ascertain an angle given a point on a circle (Van Brummelen, 2020). As per the standards outlined by the National Council of Teachers of Mathematics (1989, 2000), trigonometry holds a significant position within mathematics education curricula because it facilitates the connections between various mathematical concepts. This is why some scholars in the field (e.g. Van Brummelen, 2020; Weber, 2005, 2008) have stressed that a robust understanding of trigonometry is paramount, as it furnishes fundamental knowledge and principles utilised in other mathematical domains like geometry, algebra, calculus, and functions. Additionally, knowledge of trigonometry serves as a precursor for tackling advanced topics like complex numbers, limits, derivatives, and integrals (Mickey & McClelland, 2017; Siyepu, 2015).

Despite its centrality in the mathematics education school curriculum, studies conducted in different parts of the world have revealed that trigonometry is one of the most difficult topics not only for secondary school students (Dhungana et al., 2023; Fahrudin et al., 2019) but also for undergraduate students, particularly preservice teachers (Adamek et al., 2005; Maknun et al., 2022). Similarly, a study by Dhungana et al. (2023), which analysed the responses of 12 Grade 10 students from Nepal to a diagnostic test, found errors in procedural and conceptual knowledge, as well as the connection between these two types of knowledge. According to Fahrudin et al. (2019), these errors were mainly due to students' lack of understanding of trigonometric periods,

angles in different quadrants, and the inability to determine trigonometric ratios of special angles. Furthermore, Adamek et al. (2005) pointed out the deficiencies in trigonometry teaching methods and the lack of emphasis on its real-world applications.

In the context of tertiary education, a study by Koyunkaya (2016) involving nine graduate mathematics education students in the Midwestern United States found that while these students had a fundamental grasp of trigonometric concepts, they encountered difficulties with flexibility and relational understanding. These challenges in understanding hindered their ability to apply trigonometric concepts effectively in diverse contexts. Similarly, a study by Walsh et al. (2017) on 50 prospective teachers at an Irish tertiary institution uncovered significant gaps in their Subject Matter Knowledge (SMK)[1] of trigonometry. The study by Nabie et al. (2018) involving 119 second-year mathematics and science preservice teachers from two Ghanaian colleges of education revealed that more than half struggled with constructing and reconstructing mental structures necessary for a deep understanding of trigonometry. This difficulty was compounded by their perception of trigonometry as abstract, challenging, and unengaging.

Study context and rationale

In the first year of the mathematics teacher education programme at the institution where this study took place, trigonometry is taught as a continuation of high school content, particularly the addition of radian measure and half-angle formulae. High school trigonometry requires students to familiarise themselves with concepts like trigonometric ratios, trigonometric identities, reduction formula, compound angles, two-dimensional and three-dimensional problems, trigonometric equations, and the three rules, namely sine, cosine and area (Department of Basic Education, 2011). Most of the aforementioned trigonometry concepts can be fairly understood by both high school and preservice teachers procedurally and conceptually, except trigonometric equations.

This study focused on preservice teachers' conceptualisation of trigonometric equations because solving them requires making connections across various trigonometric concepts and other mathematical topics (Adhikari & Subedi, 2021; Maphutha et al., 2023). The network theory by Mowat and Davis (2010) postulates that mathematical ideas do not develop in isolation but are integrated through linkages. Solving trigonometric equations is thus a central concept that connects to functions, algebra, and trigonometry itself. Trigonometric equations connect to functions by interpreting graphs and functions like quadratic, hyperbolic, linear, etc. Trigonometric equations are linked to algebra by means of manipulations like factorisation and rationalisation, and equations like linear, quadratic, cubic, and so on. Within trigonometry itself, trigonometric equations involve concepts like reciprocal, square, and quotient identities, as well as reference angles, special angles, and quadrants (Department of Basic Education, 2011).

Despite the comprehensive coverage of trigonometric equations in high school, students continue to face significant challenges in solving them, particularly in South Africa. For instance, Chigonga (2016) reveals that trigonometric equations are often perceived as complicated. Similarly, diagnostic reports from South Africa's national Mathematics Paper 2 examinations indicate that students struggle with the necessary skills and knowledge for solving them (Department of Basic Education, 2020, 2021). The varied algebraic representations of trigonometric equations often appear as meaningless symbols to students who lack an understanding of concepts such as trigonometric identities, algebraic manipulations, and angle relationships (Mutodi, 2017). According to Rohimah and Prabawanto (2019), students face difficulties in deciphering problem formats, using standard trigonometric equations, and managing mathematical manipulations. As a result, many students fail to master the mental constructions necessary for solving trigonometric equations, leading to underachievement. Research has shown that a student's mastery of mathematical concepts at an appropriate conceptual level is closely linked to their academic success (Bağ & Karamık, 2024). This study, therefore, aimed to analyse the mental constructions preservice teachers develop as they learn to solve trigonometric equations.

The research questions for this study were:

- *What mental constructions can be conjectured so that preservice teachers can develop an object conception of solving trigonometric equations?*
- *Which conjectures can be inferred from preservice teachers when they are involved in solving trigonometric equations?*

Addressing these research questions is anticipated to yield insights into the necessary steps for ensuring that preservice teachers are adequately equipped with the relevant knowledge before they commence their mathematics teaching careers. This assertion affirms the importance of establishing the readiness of prospective teachers to teach mathematics. It is argued that this readiness should be assessed at the entry (Mukuka & Alex, 2024a) and exit (Tatira, 2020) points of mathematics teacher education programmes. Such a dual assessment is assumed to ensure a comprehensive evaluation of the preservice teachers' mathematical knowledge and ability to effectively impart it to their learners.

Literature review

A gap has been noticed in the research landscape regarding the understanding of trigonometric equations among high school and preservice teachers, particularly in the context of the South African setting. This study aimed to bridge this gap by exploring preservice teachers' understanding of trigonometric equations at a higher learning institution in the

1.Subject Matter Knowledge (SMK) refers to the deep understanding and knowledge that a teacher has about the specific content they are teaching, in this case, trigonometry.

Eastern Cape province of South Africa. The research question guiding this study is: *What mental constructions do preservice teachers make when solving trigonometric equations, and how do these constructions relate to the development of an object conception of solving such equations?*

Previous research has demonstrated the utility of the APOS theory in understanding mental constructions in trigonometry. For example, Martínez-Planell and Cruz Delgado (2016) used APOS theory to investigate engineering students' mental constructions in solving trigonometric equations, specifically focusing on the unit circle approach. Their study highlighted the process and object mental constructions involved in basic trigonometric ratios and equations. Although students could solve more straightforward problems aligned with the conjectured transformations, more than 50% struggled with equations like $\sin x = \frac{1}{3}$, revealing unexpected mental constructions. This finding suggests that the genetic decomposition (GD) could be revised, though their study did not explore this. The present study builds on this by using the APOS theory to analyse first-year students' understanding of solving trigonometric equations, focusing on conjectured mental constructions at the object conception level.

The theoretical analysis of a mathematical concept to develop conjectures on students' mental constructions is central to the APOS theory. This analysis leads to the GD (Dubinsky & McDonald, 2001), which guides the data analysis of students who have learned the concept through any instructional approach. The second use of theoretical analysis is to design instructional activities that help students develop the mental structures outlined in the GD. Although this study did not engage in APOS instruction, it focused on data collection directly. A conceptual paper by Şefik et al. (2021) reviewed 125 articles focused on APOS theory, revealing that 74% of the studies used APOS theory for data analysis. The meta-analysis also showed that while the GD is useful for implementing APOS instruction and analysing data, 62% of the studies did not use it. Instead, these studies applied the theory primarily to describe data by positioning students' understanding of specific mathematical concepts according to the mental structures of the theory. Thus, studies on APOS theory vary in their application. The GD for this study is presented in the next section.

Some APOS studies focus on high school students' conceptualisation of mathematical concepts. For instance, Bağ and Karamık (2024) examined Grade 12 students' understanding of trigonometric functions using APOS theory. Data from evaluation tests and interviews with 29 participants revealed that students operated at all levels of APOS theory, with those understanding trigonometry succeeding in solving related problems. Solving trigonometric equations relies on properties of triangles, the *Pythagorean theorem*, and values of special angles, all well covered in the secondary school curriculum. Leškovski and Miovska (2020) reviewed basic trigonometric equations that high school learners can solve, such as $\sin x = \cos\frac{x}{2}$ and $\sin x = \sqrt{3}\sin\frac{x}{2}$.

They emphasised that defining trigonometric ratios and values of special angles can make learning interesting and innovative. Similarly, Rohimah and Prabawanto (2019) identified difficulties high school students face in solving trigonometric equations and proving identities. Their study, involving achievement tests and interviews, found common difficulties in understanding problems and manipulating equations, such as $2\sin^2 x = \sin x + 1$, with factorisation being a common technique.

High school students' frequent mistakes when solving trigonometric equations include challenges in understanding the questions and errors in transforming them (Usman & Hussaini, 2017). In a study conducted by Fahrudin et al. (2019) involving 203 Grade 11 students from an Indonesian high school, it was discovered that many students committed errors in concept, strategy, and calculation when solving trigonometric equations. Basic equations encountered in high school can be solved using algebraic or graphical strategies, or more specifically, a mixture of both. A study by Mutodi (2017) explored the preferred choice for Grade 12 students when solving trigonometric equations. After an analysis of students' written responses to a test and focus group interview transcriptions, Mutodi concluded that students preferred the algebraic method over the graphical one. Students felt they gained more mathematical skills by executing algebraic manipulations, and their teachers were also biased towards the algebraic method. Mutodi used a framework developed by Moschkovich et al. (1993), which describes two perspectives of a mathematical concept, treating trigonometric equations as processes and objects. The process view regards equations as rules for computation, while the object view regards equations as objects upon which further actions can be performed. In another exploration of Grade 11 students' understanding of trigonometric equations, Maphutha et al. (2023) used Mowat's (2008) theory of mathematical connections in which nodes and links are used to connect mathematical concepts. Nodes represent the main concepts in solving trigonometric equations, such as functions, algebra, and trigonometric ratios. Like a spider's web, links show how concepts relate to one another, either for causality or mutually. Factorisation and types of equations, reference angles, and special angles connect algebra and trigonometry nodes, while types of functions and their graphical representations link trigonometry and function nodes. By mapping out such relationships, a mathematical connection helps visualise variables, guide problem-solving and deepen understanding. After analysing data from task-based interviews, it was shown that although students could make algebraic connections to solve trigonometric equations, they could not make intra-trigonometry connections (Maphutha et al., 2023). Learners' inability to make intra-mathematics connections is likely caused by a lack of depth in a mathematical concept, which is solving trigonometric equations in this case.

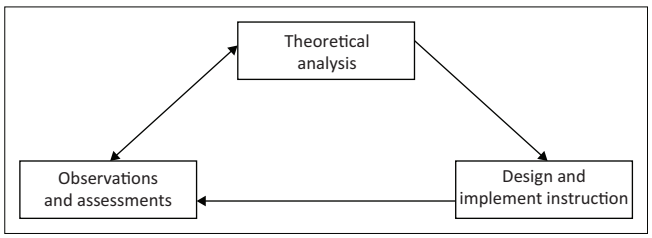
Following the literature review above, it was deemed appropriate to investigate preservice teachers' understanding of solving trigonometric equations through the lens of the APOS theoretical framework.

Theoretical framework

This research utilises the APOS theory to evaluate preservice teachers' understanding in solving trigonometric equations. This theory posits that an individual's understanding of mathematical concepts evolves through experiences that entail solving mathematical problems within a social context (Dubinsky, 1984). It suggests that individuals construct or reconstruct mental structures and organise them into schemas to address problems. Oktaç et al. (2019) affirm that APOS theory is rooted in Piaget's reflective abstraction. For this reason, APOS theory provides a constructivist perspective for understanding the construction of mathematical knowledge, particularly in higher education settings (Arnon et al., 2014).

Turning our attention to the specifics of the theory, Dubinsky and Mcdonald (2001) proposed that the APOS theory outlines the progression of understanding a mathematical concept. Initially, it is perceived as an action consisting of clear, sequential instructions to manipulate physical entities. With repetition and contemplation, these actions may transform into processes, which are mental structures capable of executing operations without the need for physical manipulation. Over time, these processes can be condensed into objects, allowing individuals to manipulate the concepts directly, such as conducting algebraic manipulations on non-standard trigonometric equations. The highest level of the APOS theory is the schema, which refers to an organised collection of actions, processes, and objects that are linked to represent a more complex mathematical concept. The APOS theory consists of three stages, which can be cyclic, namely theoretical analysis, design and implementation of instruction, and collection of data (shown in Figure 1).

Research commences with a theoretical review of the concept to be taught in terms of what is to be understood and how the understanding can be constructed by a generic student. The theoretical analysis, which forms part of the GD, is based on the researchers' knowledge of the mathematical concept and related research. The theoretical analysis informs the next stage of the design and implementation of instruction (Asiala et al., 1997). This stage involves the researchers developing activities such that if they are successfully done, students will construct the mathematical knowledge at all the levels of action, process and object. The activities before class, whole-class discussion and exercises after class are repeatable cyclically until students construct knowledge. Implementation of instruction provides an opportunity for the last stage of data collection and analysis. The common tools for data collection are tests, semi-structured interviews and task-based interviews. If the students did not attain the expected mental constructions in the findings, then the theoretical analysis is revised, triggering another APOS cycle.



Source: Asiala, M., Cottrill, J., Dubinsky, E., & Schwingerdorf, K. E. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behaviour, 16*(4), 399-431. https://doi.org/10.1016/S0732-3123(97)90015-8

FIGURE 1: Components of the APOS theoretical framework.

The arrows connecting theoretical and data analyses describe a situation where APOS theory is used as an evaluation tool for students who would have been taught in non-APOS instruction, for example traditional instruction (Arnon et al., 2014, p. 106). This study used the APOS theory as an evaluative framework to assess the preservice teachers' comprehension of solving trigonometric equations post instruction. This approach did not involve using the theory to explain the process of knowledge construction or how learners build their understanding. Arnon et al. (2014) remark that:

> [*T*]he APOS theory serves as an evaluative framework as individuals are observed in problem situations in which the researcher attempts to describe their level of understanding as well as the mental structures at work in their learning of the concept. (p. 189)

In the context of the topic at hand, APOS theory can explain the cognitive processes students undergo when solving trigonometric equations. For instance, when solving a trigonometric equation such as $\sin x = 0.5$, the action could be isolating the variable on one side of the equation, that is, changing the subject of the formula to x so that $x = \sin^{-1}(0.5)$. Solving a basic trigonometric equation to find the reference angle describes an action conception. The equation $\sin x = 0.5$ is relevant to solving trigonometric equations; students manipulate complex trigonometric equations to reduce them to this form. Students solve simplified equations, which is common at the secondary school level in South Africa (Department of Basic Education, 2011). When the action is repeated and internalised, it becomes a process. The individual can now imagine the action in their mind without performing it. In this example, the process could be understanding that the inverse sine function can be used to solve for x in the equation $\sin x = 0.5$.

At the process stage, students appreciate the infinite nature of solutions of trigonometric equations; if the interval is not given, students need to find the general solution, or else they find the values of the angles that satisfy the equation within a given interval (Mutodi, 2017). Once thoroughly understood, a process can be incorporated into an object. This means that the process can be used as an entity on its own. Solving non-standard equations where further actions and processes can

be performed using algebraic manipulations to reduce the equation to a basic format is an example of an object-level conception. Finally, the schema could be the general method for solving trigonometric equations, which involves isolating the variable, applying inverse trigonometric functions, and finding the solution set within the specified interval. The GD illustrated in Figure 2 gives the researchers' conjectures of the actions, processes, and objects students need to develop the schema for solving trigonometric equations.

The GD in Figure 2 describes the mental structures and mechanisms at play to develop the schema for solving trigonometric equations. Students start constructing mathematical knowledge at the action level. The highest level is the object, which is the most difficult to attain (Bilondi & Radmehr, 2023). At this stage, students have combined object conception with previously held schemas to develop the current schema, as shown in Figure 2. Overall, the coherent collection of actions, processes, and objects constitute the schema shown by the outside boundary.

Notably, this theoretical framework has been employed to analyse preservice teachers' understanding of various mathematical concepts within South Africa and beyond. For instance, Martínez-Planell and Delgado (2016) provide empirical evidence supporting the utility of the APOS theory in analysing the mental constructs students form when developing a unit circle approach to sine, cosine, and their corresponding inverse trigonometric functions. Similarly, Siyepu (2015) utilised the APOS theory to scrutinise students' errors in calculating trigonometric function derivatives. Similarly, Nabie et al. (2018) employed the APOS theory to examine preservice teachers' knowledge and perceptions of trigonometric concepts. Ngcobo et al. (2019) applied the APOS theory to comprehend the mental constructs students

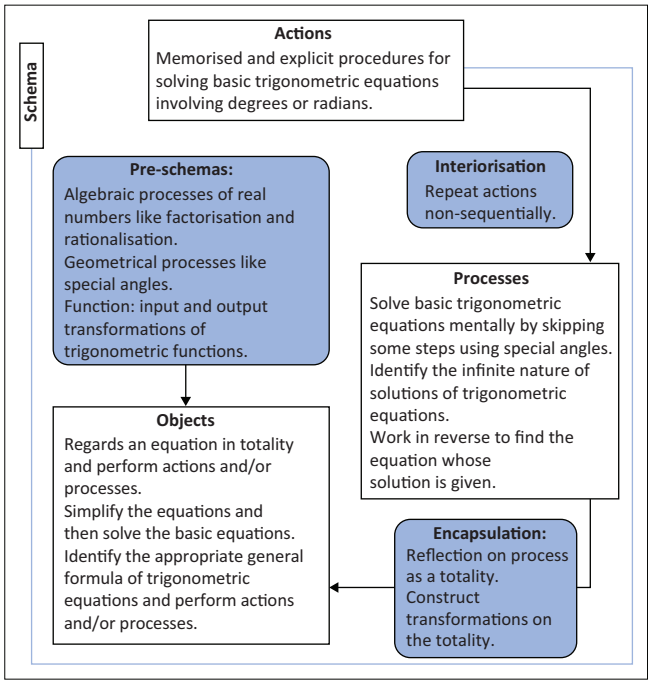


FIGURE 2: The genetic decomposition for solving trigonometric equations.

form when learning trigonometry, using these insights to propose enhancements in teaching methodologies. Furthermore, Walsh et al. (2017) leveraged the APOS theory to understand the mental constructs that second-level teachers form when learning trigonometry. They utilised those insights to ascertain the subject matter knowledge required for effective trigonometry instruction.

Besides the concepts related to trigonometry, the APOS theory has been instrumental in analysing preservice teachers' understanding of other mathematical concepts, such as the application of elementary row operations in solving systems of linear equations, student teachers' understanding of binomial series expansion (Tatira, 2021, 2023), decimals with a recurring single digit (Burroughs & Yopp, 2010), Cramer's rule (Ndlovu & Brijlall, 2015), eigenvectors (Salgado & Trigueros, 2015), derivatives (Moru, 2020; Tatira & Mukuka, 2024), inverse matrix method (Kazunga & Bansilal, 2020), and vector spaces (Mutambara & Bansilal, 2019), among others.

These studies collectively affirm that the APOS theory can guide the development of teaching strategies that foster a deeper understanding of trigonometry by facilitating the transition from actions to processes and from processes to objects, ultimately leading to a coherent schema. However, we are unaware of a study that has yet to explore preservice teachers' understanding of solving trigonometric equations using the APOS perspective, particularly in South Africa. Therefore, this study not only augments the existing body of knowledge on preservice teachers' understanding of trigonometric equations but also probes the applicability of the APOS theory within the South African milieu. By evaluating preservice teachers' knowledge levels regarding solving trigonometric equations, this study is poised to offer valuable insights into enhancing the quality of mathematics education for future teachers and their future learners.

Research methods and design

Research design

We employed a qualitative case study design to address the research questions posed. This approach is particularly effective for exploring and understanding complex phenomena within their real-life context (Yin, 2009). Our study focused on a single case involving a group of first-year preservice teachers enrolled in a Bachelor of Education (BEd) Programme, specialising in teaching mathematics and either physical or life sciences.

Although our study did involve quantifying frequencies for the categories shown in Table 2 – Table 4, most of our analysis was qualitative. We examined the actual test scripts to identify preservice teachers' understandings and the levels of the APOS framework they attained. Combined with existing literature (Leedy & Ormrod, 2019; Rohimah & Prabawanto, 2019), this type of analysis offered insights into potential teaching strategies or interventions that could enhance preservice teachers' ability to solve trigonometric equations.

According to Yin (2009), a case study design is justified when the research aims to answer 'how' and 'why' questions, involves an in-depth investigation of a contemporary phenomenon within its real-life context, and when the boundaries between the phenomenon and context are not evident. Our research meets these criteria as it seeks to find out how preservice teachers develop their understanding of solving trigonometric equations.

Participants and data collection

Data were collected through a formal individual test administered to 223 first-year preservice teachers enrolled in the BEd Programme at a South African university to investigate the two research questions. All participating students were exposed to traditional instruction delivered in a lecture format on trigonometry. While the original test had many questions on different aspects of trigonometry, this analysis reports explicitly on two items related to trigonometric equations (Martínez-Planell & Delgado, 2016). We selected these two items because each required a long, multi-step solution. Table 1 illustrates the specific items whose solutions were analysed.

Item 1 requires students to present solutions in radians, whereas Item 2 requires solutions in degrees. The goal was to see students' solutions in both measures. Regarding Item 2, it is worth noting that the constant k in the equation $4\sin^3 y - 8\sin^2 y - \sin y + k = 0$ was to be substituted with 2. This value should have been derived from a preceding question where students were asked to find the value of k given that the polynomial $f(x) = 4x^3 - 8x^2 - x + k$ is divisible by $2x - 1$. This part of the question, not being part of the solving trigonometric equations, was omitted in this study.

Following the initial analysis of the test scripts, eight students were purposefully chosen from the original group of 223 for open-ended interviews. The interviews were designed to be open-ended, with questions tailored to the specific solutions provided by each interviewee. These interviews aimed to probe the preservice teachers' understanding of trigonometric equations and the reasoning behind their solutions (Douglas, 2022). Insights gleaned from these interviews were subsequently cross-verified with the initial analyses of the written solutions as presented in the results section. Like the procedure employed by Martínez-Planell and Delgado (2016) and Tatira (2023), this process helped identify the cracks in students' understanding. It reinforced our comprehension of the APOS levels at which the preservice teachers functioned. Consequently, the transcriptions of these interviews constituted the second data set for this study.

TABLE 1: Administered test items on trigonometric equations.

Item	Solve the equation
Item 1	$\frac{1}{\tan^2\theta} = 3$, for $0 \leq \theta \leq 2\pi$ giving the answer in radians in terms of π
Item 2	$4\sin^3 y - 8\sin^2 y - \sin y + k = 0$ for $0° \leq y < 360°$

Note: For item 2, k was to be replaced with 2.

Data analysis

Initially, the solutions were categorised as blank, incorrect, partially correct, and correct, and the frequency of each category is shown in Table 2. Subsequently, we conducted a content analysis of the written solutions to identify the level of mental constructions made and discern the various elements of the APOS theory present in all solution strategies, whether at the action, process, or object level. To provide more context to the predefined categories, excerpts from the participants' answer scripts were selected and presented. Furthermore, we performed a thematic analysis of the interview transcripts to detect and discuss emerging patterns in the data. As earlier indicated, interview data analysis helped identify the mental constructions made and reinforced our understanding of the APOS theory levels at which the preservice teachers were functioning. Upon examination of both the written solutions and interview responses, we interpreted their significance and characterised the students' operational level in the context of the APOS theory.

Ethical considerations

In adherence to ethical standards, the confidentiality of the participants was maintained. We did this by ensuring no participant was identified by their names. Instead, pseudonyms (A1 to A223) were used as participants' identities for reporting purposes. Each participant's informed consent was also sought at the start of the study, and they were free to withdraw at any point of the research process if they felt threatened and unsafe. Ethical clearance to conduct this study was obtained from the Walter Sisulu University Faculty of Educational Sciences (FEDS) ethical committee (No. FEDSRECC01-12-20).

Results

Table 2 illustrates the participants' responses to two items on trigonometric equations. The responses varied from blank (no attempt made) to correct (correct solutions given). The data in Table 2 suggest a negligible difference between Item 1 and Item 2 regarding partially correct and correct solutions.

The frequency distribution in Table 2 shows that more participants left Item 2 unanswered ($n = 11$) compared to Item 1 ($n = 1$), possibly due to a lack of confidence in factorising the cubic polynomial in Item 2. The number of incorrect responses was higher for Item 1 ($n = 20$) than for Item 2 ($n = 6$). However, when considering a combination of blank and incorrect responses, the difference is minimal, suggesting similar performance levels across both items.

TABLE 2: Distribution of participants' responses to two items on trigonometric equations.

Response category	Item 1 frequencies	Item 2 frequencies
Blank	1	11
Incorrect	20	6
Partially correct	170	169
Correct	32	37
Total	**223**	**223**

Item 1 analysis

As per the data presented in Table 2, a small proportion of participants (n = 32; 14.3%) achieved correct solutions for Item 1. This suggests that most participants (n = 191; 85.7%) did not fully succeed in Item 1. Table 2 further reveals that one participant did not attempt the question, while 20 participants (accounting for 8.97%) answered it incorrectly. The majority of participants (n = 170; 76.2%) managed to solve the equation partially.

Upon analysing the solutions of participants who answered Item 1 incorrectly, it was observed that most of them attempted some form of algebraic manipulation. They tried to work directly with the tangent function or convert it into a sine to cosine ratio. However, they struggled to determine the correct reference angle. This difficulty stemmed from their insufficient understanding of determining the trigonometric ratios of special angles.

This lack of understanding is exemplified in the solution provided in Figure 3 by A129. This participant incorrectly determined the value of θ as $\frac{\pi}{2}$ instead of $\frac{\pi}{6}$.

It is also evident that this participant overlooked $\tan\theta = -\frac{1}{\sqrt{3}}$, which could have led to additional values of θ in the quadrants where the tangent function is negative. This highlights a common area of misunderstanding among this group of participants.

In relation to the participants who achieved partially correct solutions, a detailed analysis of their test scripts unveiled a variety of approaches. These distinct methods have been classified into six categories, as depicted in Table 3.

A notable observation from the results in Table 3 relates to the combined responses from categories 2 and 3. It shows that 54 participants, representing 31.8%, could identify a reference angle and an additional angle in the second or third quadrant. This suggests that these participants understand how trigonometric ratios function in different quadrants. However, these participants did not generate a complete solution. They either overlooked the quadrants where a specific trigonometric function is negative or focused solely on the first two reference angles, failing to acknowledge the periodic nature of trigonometric functions. Another noteworthy point is that most participants did not attempt to derive general solutions. Although deriving general solutions was not necessarily a requirement, it can be beneficial for determining specific values within a given interval.

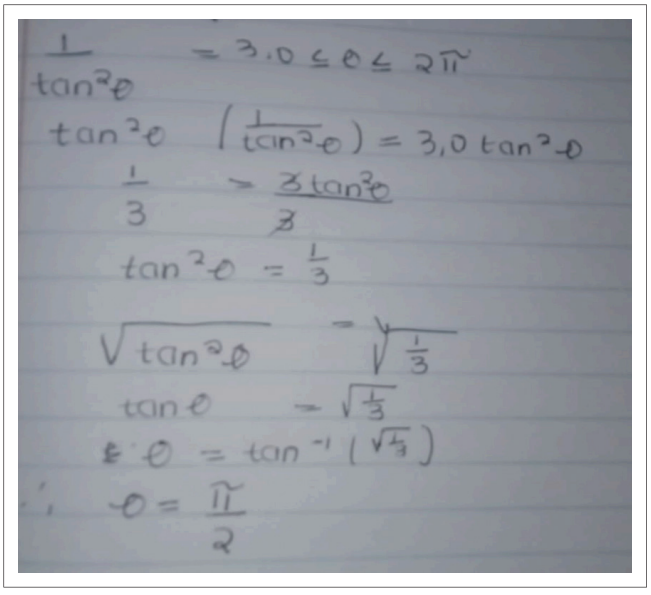


FIGURE 3: Example of an incorrect solution (Participant A129).

Figure 4 presents an example of a solution from a participant identified as A192. This participant was able to identify the reference acute angle and its corresponding value in the third quadrant, where the tangent function is positive. This highlights a common level of understanding among this group of participants.

The solution in Figure 4 demonstrates some reasonable grasp of the reduction formula and the conversion process from degrees to radians. However, the participant did not identify the angles in the second and fourth quadrants, where the tangent function is negative. This highlights an area for improvement in their understanding of the behaviour of the tangent function across different quadrants.

Another notable proportion of participants who obtained partially correct solutions fell into Category 1 (n = 49; 28%). These participants only managed to determine a reference

TABLE 3: Distribution of partially correct solutions for Item 1 (N = 170).

Category	Description	Frequency (n)	%
1	Found one reference angle in degrees or radians (30° or π/6)	49	28.8
2	Identified two reference angles, π/6, and 5π/6.	29	17.1
3	Identified π/6 and third quadrant angle 7π/6	25	14.7
4	Found solutions in the first three quadrants	11	6.50
5	Correct reference angles but confused degrees and radians	22	12.9
6	Derived general solutions but missed the specified interval	34	20.0

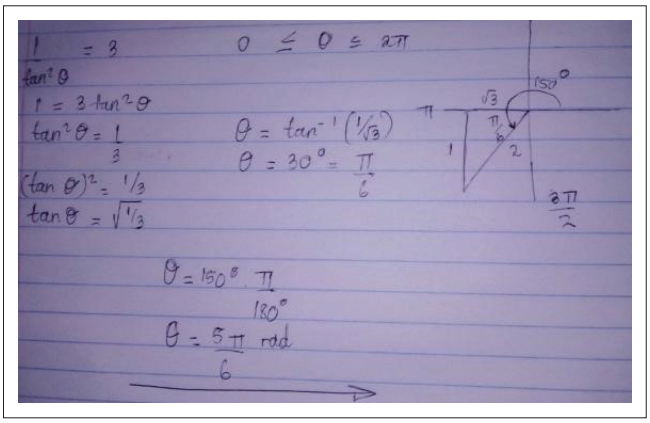


FIGURE 4: Partially correct solution with values in quadrants where tangent is positive (Participant A192).

acute angle, without further identifying other solutions within the specified interval. The solution provided by participant A105, as shown in Figure 5, provides evidence of this category of participants.

Additionally, it was observed that some participants in Category 1 did not convert the angle measure from degrees to radians. This suggests a lack of understanding among these preservice teachers about reference angles, their relationships with angles in different quadrants, and the concept of radians as a unit of angular measure. It also indicates a difficulty in converting between radians and degrees.

Table 3 also indicates that 34 (20%) participants who produced partially correct solutions fell into Category 6. These participants correctly identified the reference angles and generated general solutions but failed to restrict their solutions to the specified interval. Figure 6 presents a solution from a participant identified as A150, providing a representative example of Category 6 for partially correct solutions. This highlights a common area of misunderstanding among this group of participants.

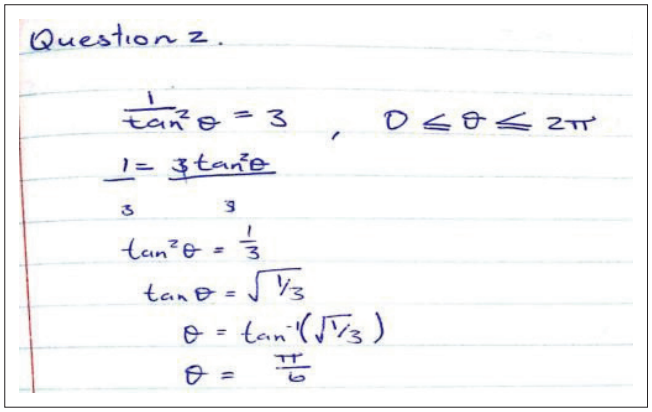


FIGURE 5: Partially correct solution with a single reference acute angle (Participant A105).

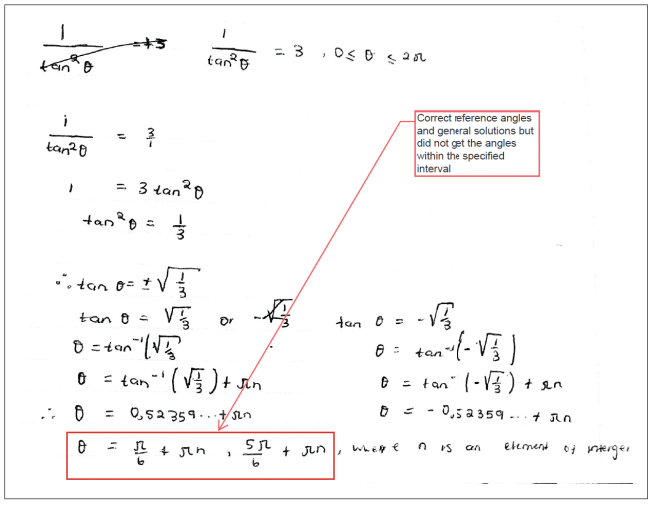


FIGURE 6: Solution with correct reference angles and general solutions (Participant A150).

Based on Figure 6, it is evident that participants in Category 6 did exhibit a strong understanding of solving trigonometric equations. However, their oversight of the specified interval in the question could indicate two things. The first one is that they might have missed or overlooked the specified interval in the question. This could be due to rushing through the question or not reading the question thoroughly. The second one is that they might not fully understand the importance of the specified interval in determining the specific solutions for a trigonometric equation. Hence, there was no coherence in their understanding, meaning that the schema stage was not attained.

Category 4 comprised those who got three solutions in the first, second, and third quadrants but ignored the fourth quadrant angle. On one hand, it might be an oversight on the part of the students, where they missed or forgot to consider the fourth quadrant. On the other hand, it could also indicate a misunderstanding that trigonometric functions are periodic in nature, meaning they repeat their values in regular intervals or periods.

Category 5 comprised the participants who got the correct reference angles in degrees but mixed degrees and radians when generating the general solution. This group of participants did not generate specific solutions within the given interval but ended up with a general solution such as the one displayed in Figure 7 by a participant identified as A67.

As noted from Figure 7, the issue of mixing up degrees and radians in the general solution of a trigonometric equation could be due to the fact that the student did not fully understand the difference between degrees and radians, and when to use each. Converting from degrees to radians is an action conception which students can easily do using the relationship $\pi = 180^\circ$. Therefore, students should be made

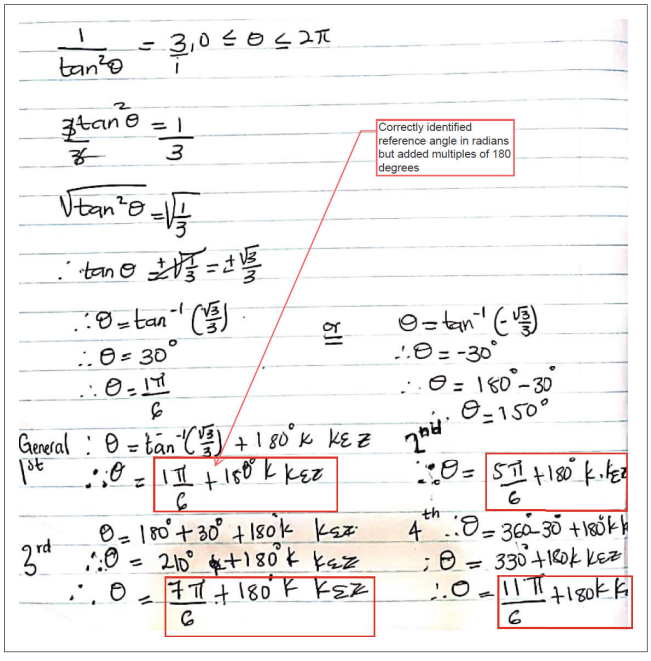


FIGURE 7: Correct reference angle but mixed units in general solutions (Participant A67).

aware that degrees and radians are just different units for measuring angles, much like inches and centimetres are different units for measuring length.

Item 2 analysis

Analysis of Item 2 scripts showed that most participants who left the question blank (n = 11; 4.9%) only copied it without attempting it. This was mainly attributed to students' failure to derive the value of k in the preceding question. This was visible in the scripts for those who got the question wrong (n = 6; 2.7%). This group of students struggled with this question as they could not substitute $k = 2$, which they had found in the previous sub-question to get $4\sin^3 y - 8\sin^2 y - \sin y + 2 = 0$. Figure 8 illustrates a participant's solution that demonstrates challenges in solving the equation, stemming from the oversight of not substituting k with 2.

As indicated in Table 2, most participants (n = 169; 75.8%) achieved partially correct answers for Item 2. An in-depth analysis of the solutions for Item 2 revealed various categories of partially correct responses. These ranged from participants who were unable to proceed after factorising the left-hand side of the equation $4\sin^3 y - 8\sin^2 y - \sin y + 2 = 0$, to those who successfully derived the general solutions. Table 4 provides a detailed distribution of these categories among these participants.

The data shown in Table 4 reveal that the majority (n = 75; 44.4%) of partially correct participants successfully identified the reference angles and managed to derive the general solutions but could not write down the solutions within the specified interval. This suggests these participants understood algebraic cubic equations, the sine function's behaviour across quadrants, and the trigonometric ratios of unique angles like 30°.

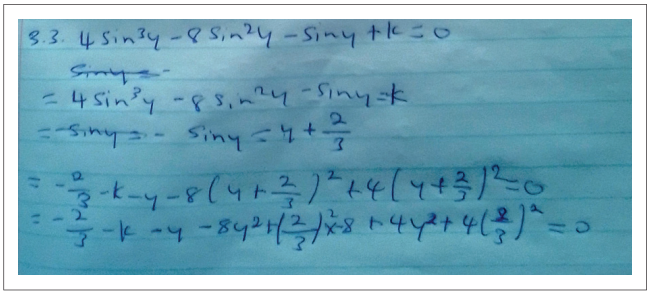


FIGURE 8: Incorrect solution due to failure to replace k with 2 (Participant A176).

TABLE 4: Distribution of partially correct solutions for Item 2 (N = 169).

Category	Description	Frequency (n)	%
1	Factorised the cubic polynomial but could not proceed	10	5.9
2	Got two solutions where sine is positive (30° and 150°)	25	14.8
3	Identified two reference angles with general solutions	40	23.7
4	Identified three reference angles with general solutions	19	11.2
5	Identified four reference angles with general solutions	75	44.4

One thing that this group lacked was paying attention to all the specified details in the question. It is also possible that some did not understand the difference between a general solution and a particular solution of a trigonometric equation. As earlier indicated, this provides evidence of the need to understand the role of each component in a question to arrive at the correct and complete solution.

The other notable observation is related to category 2. Based on the results displayed in Table 4, 25 participants (14.8%) only concentrated on the solutions associated with quadrants where the sine function is positive. This means that these participants discarded the solutions associated with $\sin y = -\frac{1}{2}$.

An excerpt presented in Figure 9 by a participant (A151) is typical of this group of participants.

The solution excerpt presented in Figure 9 suggests a gap in knowledge about the periodic nature of the sine function and how it can take both positive and negative values. As shown in Table 4, 10 students (5.9% of partially correct responses) successfully factorised the cubic polynomial but struggled to determine angle y within the given interval. Their difficulties seemed more related to trigonometry than algebra, possibly due to a limited understanding of inverse trigonometric functions, unfamiliarity with special angles, or misconceptions about finding all possible solutions. In the specific example, participant A80 (as shown in Figure 10) was able to factorise the equation correctly but made a mistake in identifying the reference angle. Instead of recognising the correct reference angles of 30° and 210° (or 30° and –30°) that are associated with the equation $\sin y = \pm\frac{1}{2}$, the participant found 45°, which is incorrect. Nevertheless, this student had good construction of knowledge of solving trigonometric equations, with skills to reject $\sin y = 2$ as a possible solution.

The excerpt displayed in Figure 10 suggests a gap in student's understanding of how to correctly determine reference angles in trigonometric equations.

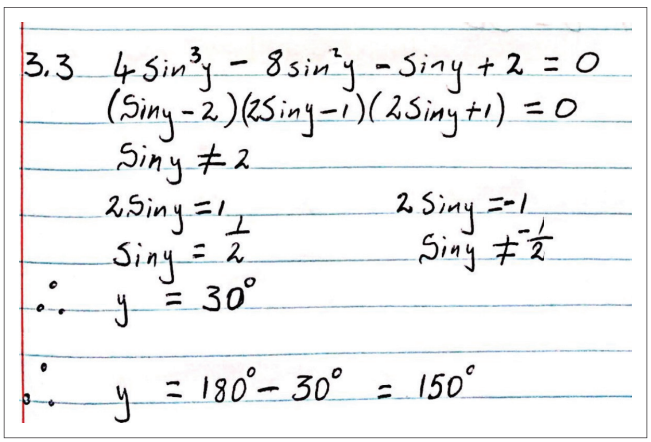


FIGURE 9: Example of a solution typically focusing on quadrants where sine is positive (Participant A151).

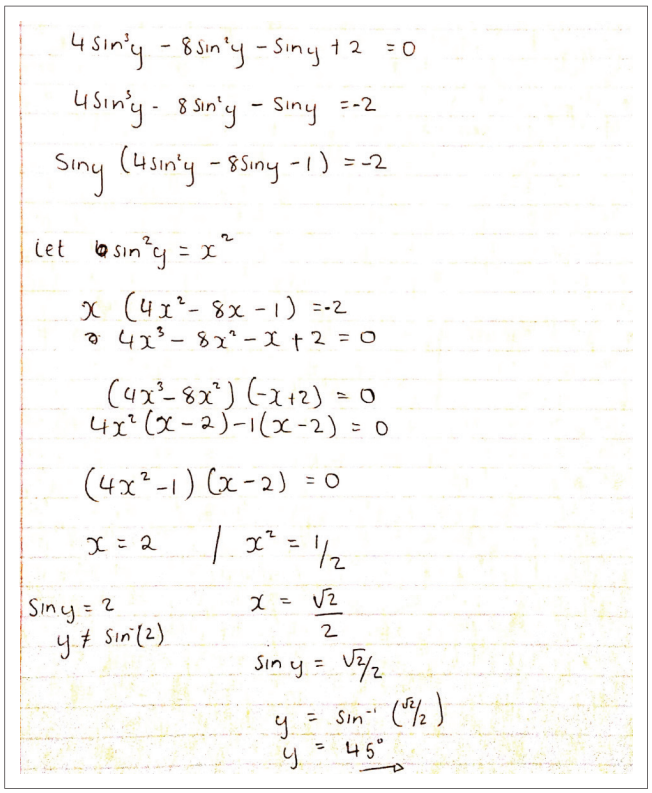


FIGURE 10: Solution with a misidentified reference angle (Participant A80).

Participants' attained APOS levels in task performance

Data analysis from items 1 and 2 reveals that preservice teachers demonstrated a range of competencies within the APOS theoretical framework.

First, it has been observed that 1 participant and 11 participants did not attempt Item 1 and Item 2. Instead, they only copied the question or left the space blank. This suggests a potential lack of confidence or uncertainty about the foundational concepts necessary to solve such trigonometric equations. The interview with Participant A215 exemplifies this:

Researcher:	I noticed you did not answer Item 2; you only copied down the question. Can you tell me why?
Participant A215:	The powers on sine confused me, and I wasn't sure where to start from.
Researcher:	Okay, I see, if you were presented with the equation $4x^3 - 8x^2 - x + 2 = 0$, how would you approach it?
Participant A215:	I think I could look for factors first.
Researcher:	And how would you determine those factors?
Participant A215:	I'm sorry, I have forgotten how to do that.
Researcher:	Do you see any connection between this equation and the one you were given in Item 2?
Participant A215:	There seems to be a relationship, but I need to revise first for me to answer this question.

The reluctance to engage with Item 2, as highlighted by the response from Participant A215, indicates a disconnect at the Action stage of the APOS theory, where fundamental procedures for addressing such problems have not been solidified.

Second, in their attempts to solve the two trigonometric equations, some preservice teachers made errors indicative of conceptual misunderstandings. For instance, during Item 1, incorrect transformations were made from the tangent to other trigonometric functions. Errors on Item 2 largely involved flawed algebraic manipulations, such as failure to determine the value of k and substitute it in the given equation and failure to factorise a cubic polynomial. These issues were highlighted during an interview with A114, who incorrectly converted the tangent function to the secant function:

Researcher:	Did you attend the class on radians?
Participant A114:	No, I missed that class.
Researcher:	How did you approach solving the first equation?
Participant A114:	I tried to change $\tan^2\theta$ to $\sec^2\theta$ then make it the subject of the formula.
Researcher:	Where did you get $\sec^2\theta$?
Participant A114:	Because it is the derivative of $\tan\theta$.
Researcher:	Oh I see! Why did you do that?
Participant 114:	Yeah! Let me just say I don't know sir.

The dialogue with Participant A114, alongside the errors documented in Figure 4 and Figure 9, suggests a gap in the foundational understanding of trigonometry and algebra. Such errors point to a pre-Action stage within the APOS framework, where students have not yet mastered the initial steps and procedures necessary for solving such equations. Pre-Action denotes no evidence to any of the conjectured mental constructions in the GD (Martínez-Planell & Delgado, 2016).

Third, the data analysis from Item 1 and Item 2 indicates that 76% (on average) of the participants demonstrated a partial understanding, as evident in their partially correct solutions. Based on APOS theory, this suggests that these preservice teachers were likely at the Action or Process stages because they could execute certain procedures but did not achieve complete solutions. The totality of understanding the solution of trigonometric equations was not evident. A relationship exists between students' success in solving problems and their construction of mathematical knowledge (Bağ & Karamık, 2024). Notable errors included misidentifying reference angles, ignorance of the solution interval, and confusion between radians and degrees. In Item 2, we observed that some participants stalled after decomposing the cubic polynomial into factors. This suggested a need for further development in their object conception for solving trigonometric equations. Many students did not fully consider all quadrants or the specified interval. This hints at an incomplete Object stage comprehension, where a holistic

understanding of concepts is expected. These findings are exemplified in the solution provided by a participant identified as A42, as displayed in Figure 11.

While several omissions had been made in the solution presented in Figure 11, one of the things that caught our attention was the student's decision to divide each term in the interval by 2 to get $0° \leq 15° \leq \pi$. An interview with this student sheds light on their reasoning:

Researcher:	You found one reference angle, but then you divided the interval by 2, which resulted in $0° \leq 15° \leq \pi$. Can you explain why?
Participant A42:	After determining that the angle was 30°, I divided the interval by 2 to express the answer in terms of π, since the question says give the answer in terms of π.

As indicated earlier, this pattern suggests that while this group of students may have reached the Action or Process stages of the APOS theory, they lacked the comprehensive Object stage. Their ability to execute certain steps correctly was overshadowed by confusion over specific components of the questions, leading to incomplete or incorrect solutions. For example, the decision by the student to divide the interval by 2, resulting in $0° \leq 15° \leq \pi$, highlights a misunderstanding of the relationship between radians and degrees, as well as a misapplication of mathematical procedures. This indicates that such participants could identify and use certain mathematical operations. Nevertheless, their conceptualisation did not extend to the Object level, where the interrelationships and generalisations of mathematical concepts are fully understood and applied consistently. The lack of Object comprehension is further highlighted by the preservice teachers' inability to integrate their procedural knowledge with the conceptual demands of the tasks, leading to a fragmented understanding characteristic of the earlier stages of the APOS theory.

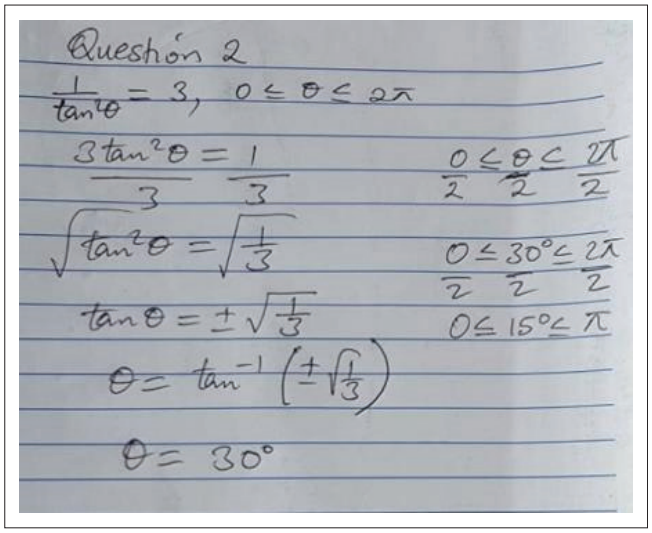


FIGURE 11: Solution showing partial correctness with conceptual gaps (Participant A42).

Fourth, it has been noted that a small proportion of preservice teachers (15.5% on average) successfully answered the questions and showcased a solid understanding of the material. Figure 12 and Figure 13 illustrate this proficiency with a set of correct solutions from a participant identified as A203.

The participant whose solutions are presented in Figure 12 and Figure 13 was then invited for an interview, which provided further insight into their understanding:

Researcher:	I have seen that you managed to solve both equations correctly. What was your strategy?
Participant A203:	I have a strong understanding of this topic, Sir. I followed the guidelines of each question, and I also applied my knowledge of quadrants, the CAST diagram, and general solutions.
Researcher:	What do you mean by CAST diagram?
Participant A203:	We use it to identify the quadrants where trigonometric functions are positive. It stands for COSINE-ALL-SINE-TANGENT, denoting that cosine is positive in the fourth quadrant, all trigonometric ratios are positive in the first quadrant, sine is positive in the second quadrant, and tangent is positive in the third quadrant.
Researcher:	Interesting! Did you use a calculator when solving these equations?

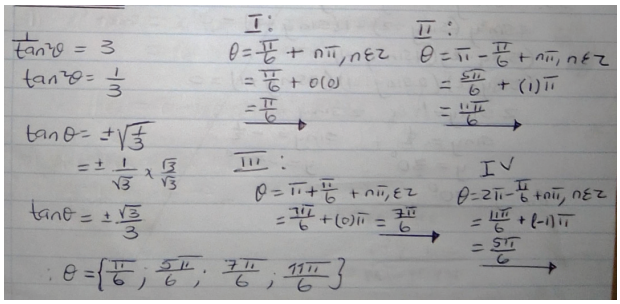


FIGURE 12: Example of a correct solution (Participant A203).

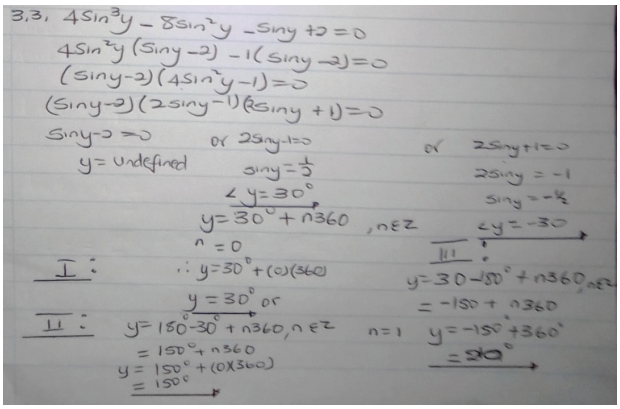


FIGURE 13: Example of a correct solution for Item 2 (Participant A203).

Participant A203:	Yes, but I only used it to find the angles … for example, the tan inverse and the sin inverse.
Researcher:	Is it possible for you to solve these equations without a calculator?
Participant A203:	Actually, yes. After writing that test, I discovered that I could use those triangles for angles like 45° and 30°.
Researcher:	Are you referring to special angle diagrams?
Participant A203:	Yes … I think so.

The conversation in the interview excerpt above and solutions displayed in Figure 12 and Figure 13 indicate that Participant A203 has likely achieved the Object level within the APOS framework, as exhibited in their use of the CAST diagram for solving trigonometric equations and their ability to find angles with or without a calculator. The participant's reference to special angle diagrams for specific angles further demonstrates their conceptual understanding of these mathematical constructs and their practical application. Attaining the three APOS levels coherently implies that this group of participants had attained the full schema development of solving trigonometric equations.

Discussion

A pivotal finding that emerged from our analysis is the low success rate among preservice teachers in solving trigonometric equations. Results show that only about 15.5% of participants correctly solved both equations. The majority grappled with algebraic manipulations, determining reference angles, understanding the relationship between angles in different quadrants, and converting between degree and radian measures. A significant lack of understanding of the periodic nature of trigonometric functions and the derivation of general solutions was also observed. These findings resonate with research conducted in other settings indicating low success rates in trigonometry among both preservice teachers and students. For instance, Fahrudin et al. (2019) found that a substantial proportion of Indonesian high school students made concept, strategy, and calculation errors in solving trigonometric equations, just like the difficulties some of the participants in this study faced.

The findings of this study provide evidence that some of the lacking mental structures in mathematics at the secondary school level persist into university, as observed in earlier studies (Mukuka & Alex, 2024a; Mukuka et al., 2020). Preservice teachers experience difficulties even in topics that they learned at the secondary school level (Malambo et al., 2018; Mukuka & Alex, 2024b). Byers (2010) identified a lack of coherence in the transition from secondary school to college mathematics as a contributing factor. In the context of this study, the backwash effects of examinations at the secondary school level emerged as another significant barrier to students' comprehensive understanding of various topics, including trigonometry. This was particularly evident during interviews with A35, who stated:

> 'In high school, we didn't fully understand these things. The focus was on making us to pass the exam. Our teacher was only teaching us how to answer questions and not the real trigonometry equations because time was not enough. In short, we were just cramming.' (Preservice teacher, female, A35)

This statement affirms the urgent need to address these issues before preservice teachers begin teaching in schools; otherwise, the vicious cycle is likely to continue. Assessing the readiness of prospective teachers to teach mathematics, both at the entry (Mukuka & Alex, 2024a) and exit points (Tatira, 2020) of mathematics teacher education programmes has also been emphasised. This emphasis has been attributed to the fact that such dual assessment is poised to provide a comprehensive evaluation of preservice teachers' mathematical knowledge and ability to effectively impart this knowledge to their students.

In this analysis, we also observed a tendency among preservice teachers to overlook the instructions or guidelines in the question, particularly the specified interval within which the solution must lie and whether the angles must be given in degrees or radians. This led to incomplete solutions and confusion between degrees and radians when generating general solutions. Akkoc (2008) reports that students prefer to use the degree rather than the radian measure. This is consistent with recent research in the field. For instance, a study by Maknun et al. (2022) found that students displayed challenges in utilising angles measured in radians and acknowledging the value of π. They also observed a pattern where students would mechanically follow the steps to convert angles from radians to degrees without deeply comprehending the foundational formulae. The research further revealed that those students had difficulties in calculating the values of trigonometric functions, especially for angles situated in various quadrants. Similarly, an exploration of preservice teachers' mathematical knowledge for teaching by Tatira (2020) and Mukuka and Alex (2024a) revealed that preservice teachers' mastery of content knowledge in trigonometry in general was inadequate. This lack of mastery could also be a factor in the confusion between degrees and radians observed in the current study.

Regarding the attained APOS levels in task performance, this study's findings reveal that a small proportion of preservice teachers, accounting for 8.5%, on average, were operating at the pre-Action stage of the APOS framework. This group mainly comprised those who left the questions unanswered and those who generated incorrect solutions. These preservice teachers had not yet mastered the fundamental concepts and procedures necessary for solving trigonometric equations. This observation aligns with a study by Nabie et al. (2018), which established those preservice teachers often perceived trigonometry as abstract and difficult. They also found that

the teachers had limited conceptual knowledge of basic trigonometric concepts. This lack of understanding could contribute to overlooking key details in problem instructions, which was quite prominent in our analysis of preservice teachers' solutions.

On the other hand, the current study reports that the majority of preservice teachers demonstrated partial understanding, suggesting that they were operating at the Action or Process stages of the APOS theory. These participants showed a good understanding of solving algebraic cubic equations and possessed sufficient knowledge of the behaviour of trigonometric functions in different quadrants. They also understood the trigonometric ratios of special angles like 30°. However, this group often overlooked specific details in the question and demonstrated an inadequate understanding of the relationship between degrees and radians. Similarly, Martínez-Planell and Cruz Delgado (2016) found that while 80% of high school students produced partial or complete solutions for basic equations with special angles, more than 50% struggled with equations where the ratio did not correspond to a special angle. Interestingly, a small proportion of preservice teachers demonstrated a solid understanding, particularly those who answered both questions correctly (15.5%). This suggests that this group of preservice teachers had achieved the Object level within the APOS framework.

The current study's findings complement those of Ngcobo et al. (2019) by providing evidence of preservice teachers' mental constructions at different stages of the APOS framework. Despite the inadequacies exhibited by preservice teachers in solving the two trigonometric equations, the fact that the majority operated at the Action and Process levels of the APOS framework is encouraging. Furthermore, Martínez-Planell and Delgado (2016) suggest that students with a process conception of the conjectured mental constructions perform better in problem-solving activities. This provides a promising direction for future educational strategies in trigonometry. In any case, 'a concept is first conceived as an Action, that is, as an externally directed transformation of a previously conceived Object' (Arnon et al., 2014, p. 20). Students typically progress from a procedural (Action or Process) to a conceptual (Object) view of a mathematical concept (Sfard, 1991). For instance, in the present study, some preservice teachers correctly applied algebraic manipulation to isolate the trigonometric function (Action level) and determined a reference angle using inverse trigonometric functions (Process level). Subsequently, such students might be able to interiorise and encapsulate the process as they repeat and reflect on actions and processes.

Conclusion

This study highlights the need for strategic instructional interventions to enhance preservice teachers' proficiency in trigonometry, a critical area of mathematics education. The low success rates in solving trigonometric equations observed globally underscore the urgency for a pedagogical shift. Strengthening preservice teachers' foundational knowledge of mathematics is crucial, particularly in understanding the periodicity of trigonometric functions and the application of angles in both radians and degrees. A solid grasp of algebraic principles, from basic to advanced concepts, is equally important for effective teaching and problem-solving in trigonometry.

The preservice teachers in this study demonstrated various mental constructions in solving trigonometric equations, as outlined in the theoretical framework. While most participants were able to solve basic trigonometric equations using special triangles, they often faced challenges when working with more complex equations requiring algebraic manipulation. The majority of the participants operated at the Action and Process levels of the APOS framework, suggesting that their understanding was more procedural than conceptual. To tackle more advanced problems, students must connect multiple mathematical concepts within and beyond trigonometry. Future studies could refine the conceptual framework based on students' responses and incorporate the APOS framework as a developmental tool to guide instructional strategies. This approach could help design activities aimed at promoting the mental structures and mechanisms necessary for achieving a deeper understanding of trigonometric equations.

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Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

Authors' contributions

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Data availability

Due to privacy and ethical considerations, the data are not publicly accessible. However, the data underpinning the results of this study can be obtained from the corresponding author, A.M., upon request.

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