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Exploring pre-service mathematics teachers' intuitive strategies for solving open equivalence relations problems

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This article contributes to the body of research on equivalence relations by delineating the levels of sophistication and efficiency in the intuitive strategies employed by pre-service teachers (PSTs). Using an inductive approach, we analysed 1102 responses to two open equivalence relations problems, drawn from a 27-item baseline assessment, completed by 551 first-year PSTs across three cohorts (2022–2024). From the analysis, we began teasing out a conceptual framework that could explain the reasoning behind the responses. To authenticate the conceptual framework, we needed to unearth the strategies employed by PSTs, so we conducted follow-up task-based interviews with nine selected participants. These interviews provided insights into the reasoning behind the PSTs' written responses and helped illuminate the developmental progression in their understanding of reasoning when solving equivalence relations problem. In particular, our analysis of the interviews revealed three distinct stages of sophistication (quantitative, additive, and multiplicative) and four levels of efficiency in PSTs' reasoning when solving equivalence relations problems. The findings show that PSTs encountered significant difficulties with solving open equivalence relations problems, with 46% correct responses: 14% of which involved quantity-based reasoning, while 17% and 15% were additive and multiplicative-based reasoning. Interestingly, in 2023, multiplicative-based reasoning (which is the most sophisticated of the three types of reasoning), had the highest responses at 21.1%, compared to 11.4% in 2022 and 11.2% in 2024. Moreover, the strategy that involves working with a large equivalence difference (which strategy is the least efficient) was found to be the predominant strategy in the PSTs' workings within the additive-based reasoning. Thus, these interview findings provided deeper insights into the patterns observed in the written responses regarding PSTs' intuitive strategies.

Contribution: These findings provide insights into a possible teaching framework that delineates the stages of sophistication that students go through in the development of their knowledge of open equivalence relations problems**.** By introducing and classifying *open equivalence relations problems* as belonging to the *Comparative Relational* category of equivalence tasks, this study elaborates on the highest category of equivalence tasks. Also, this study offers a possible analytical framework for analysing levels of efficiency in students' reasoning when solving open equivalence relations problems. Consequently, the findings have implications for teacher training programmes and intervention studies with an interest in building sophistication and efficiency into PSTs' strategies for solving open equivalence relations problems.

Keywords: Intuitive strategies; equivalence relations; equivalence difference; additive reasoning; multiplicative reasoning; equal sign.

Introduction

Mathematical equivalence is a core concept in both arithmetic and algebra, the understanding of which facilitates the generation of appropriate solution strategies for solving equivalence relations problems (Alibali et al., 2007). A deep understanding of mathematical equivalence is rooted in the meaning of the equal sign as a relational symbol that equates two quantities on either side of it (Essien, 2009; Jacobs et al., 2007; Matthews et al., 2012). Across many studies into students' understanding of the equal sign, the resonant finding is that students tend to view the equal sign *operationally* rather than *relationally*: that is, as a prompt to carry out the arithmetic operation implicated in the task rather than as an indicator of equivalence (e.g., Jacobs et al., 2007; Powell, 2015). Also, given that students' initial encounters with the equal sign are in the context of

composing or building up numbers, students' view of the equal sign as a prompt to carry out the arithmetic operation implicated in the task has been found to persist well into middle school and beyond (Ardiansari et al., 2022; Essien & Setati, 2006).

While a great many of these studies were conducted in the United States and the United Kingdom, in recent years there has been a trickle of studies from South Africa (Essien, 2009; Essien et al., 2023; Machaba, 2017; Vermeulen & Meyer, 2017). Across these studies, the consensus view is that to move students from their default operational view to a relational view of the equal sign, teaching must be deliberately directed towards developing students' relational (sameness or dual) view of the equal sign. To this end of fostering a more relational understanding of the equal sign, which is essential for transitioning from arithmetic to algebra, McNeil (2007) highlights the importance of exposing students to diverse equation structures.

One hypothesis put forward by Donovan et al. (2022) that is of interest to this study is that the solutions to equivalence relations by students with a relational view of the equal sign would reveal greater sophistication in their solution strategies, as compared to students who hold an operational view of the equal sign. They argue that an important first step towards realising this goal is mapping 'the continuum of knowledge through which people are thought to progress for the target construct' (p. 87). So, they have developed the Construct Map for Mathematical Equivalence Knowledge to capture this progression in terms of the ability to solve equations with given core structures. They propose four levels to the Construct Map, with levels progressing from 1 to 4, where level 1 represents elementary knowledge and level 4 represents advanced knowledge of the target construct. Essentially, the progression is from an ability to solve equations with operations on the left (a + b = c), to equations with operations on the right (c = a + b), to equations with operations on both sides (a + b = c + d), and then to 'operations on both sides with multidigit numbers or multiple instances of a variable' (p. 87). For this study, the target construct was the solving of equivalence relations problems with operations (in particular, subtraction) on both sides with double-digit numbers and an unknown on both sides of the equal sign: a – _ = _ – d.

Coming into this study, we made the hypothesis that these equivalence relations problems (a – _ = _ – d) have the core structure of equivalence relations problems that are on level 4 of Donovan et al.'s (2022) Construct Map, with subtraction operations and unknowns on both sides of the equal sign providing the necessary added complexity for pre-service mathematics teachers. So, in staying with Matthews et al.'s (2011) hypothesis that the development of level 4 ability of 'solving equations and evaluating equation structures by comparing the expressions on the two sides of the equal sign' (p. 88) begins in elementary school, these equivalence relations problems should pose no major challenges to pre-service mathematics teachers. Also, we surmised that pre-service mathematics teachers' solutions to these problems should provide valuable empirical insights towards development and research in: (1) designing interventions that are informed by the intuitive strategies of first-year pre-service teachers (PSTs) across three consecutive academic years will provide us with a solid base from which to develop PSTs' range and flexibility in their use of strategies, so that they in turn are empowered to support their learners across the attainment range; and (2) developing possible theoretical elaborations or extensions to the Construct Map. Moreover, given the timed nature of the test, the dominant answers to a – _ = _ – d would likely be a and d, returned in the order in which they were given (a – a = d – d), or in the reverse order (a – d = a – d), reflecting the fundamental understanding that 'doing the same things to both sides of an equation maintains its equivalence, without needing to verify the equivalence relation with full computation' (Matthews et al., 2012, p. 88). We assumed that if the full computation route was taken, algebraic procedures would likely be used because the participants are post-matriculants. We also assumed that these algebraic procedures would either only involve additive operations (and therefore require only additive reasoning) or they would also involve multiplicative operations (and therefore require both additive and multiplicative reasoning). Given the very fact that these equivalence relations problems are open to both additive and multiplicative reasoning (Degrande et al., 2018), we have designated them 'open equivalence relations problems'.

Based on these assumptions we included two open equivalence relations problems in a 27-item baseline assessment designed to assess mathematical equivalence and number structuring knowledge. The instrument was used in a design study spanning three cohorts with a total of 551 first-year PSTs. One of the foci of the design study was on PSTs' knowledge of mathematical equivalence and number relations. From 2022 to 2024, the baseline assessment was administered as the first assessment at the commencement of the academic year as an opportunity to gain insights into the mathematical knowledge with which the PSTs are beginning their tertiary studies. With the two open equivalence relations problems yielding 1102 answers, we could begin an investigation into PSTs' intuitive strategies. The patterns in the pairs of answers provided us with a base from which to develop a conceptual framework that would enable us provide explanations regarding the strategies that PSTs used in solving these equivalence relations problems. However, before we could explain the strategies that PSTs used to tackle these equivalence relations problems, we needed to uncover them, as well as the spread and extent of sophistication in PSTs' intuitive strategies. To this end, we conducted follow-up interviews with nine PSTs from the 2022 cohorts using theoretical sampling (Glaser & Strauss, 1967) in line with the patterns of answers informing our conceptual framework and based on frequently appearing answers. This interview data enabled us to answer the following main question:

> *What are pre-service Mathematics teachers' intuitive strategies for solving open equivalence relations problems with an unknown on both sides of the equal sign?*

This main research question was further sub-divided into the following three questions:

- *What types of intuitive strategies can be inferred from first-year PSTs' answers to open equivalence relations problems with unknowns on either side of the equal sign?*
- *What kinds of reasoning can be seen in first-year PSTs' solutions to equivalence relations problems with unknowns on either side of the equal sign?*
- *What kinds of structuring can be seen in the procedures used by first-year PSTs when solving open equivalence problems with unknowns on either side of the equal sign?*

In the sections that follow, we provide a review of the literature, wherein we engage with numerical and non-numerical understandings of part-whole relationships and intuitive strategies. We then discuss how, through an inductive process, we explored and interpreted PSTs' intuitive strategies. Next, we present the six strategies that emerged from our data set and examine their hierarchies based on sophistication across all levels and efficiency within one level. These are then brought together in a conceptual framework that guides the study in which we operationalise the key notion of an *open equivalence relations problem* for this study. Finally, we consider the value and relevance of these intuitive strategies for researchers and PSTs.

Numerical and non-numerical understanding of part-whole relationships

Mathematical equivalence is an important component of the part-whole relation. According to Rittle-Johnson and Alibali (1999), mathematical equivalence incorporates three components: (1) the meaning of two quantities being equal; (2) the equal sign as a relational symbol; and (3) the idea of two sides to an equation. From a part-whole perspective, these three points taken together capture the idea that, with the equation seen as a single structure, differences on either side of the equal sign represent the parts that must be equivalent.

The part-whole concept is central to mathematical understanding, particularly in early childhood education, as it forms the basis for various mathematical operations (Carpenter & Moser, 1983; Resnick, 1989), including equivalence relations. Its application in both numerical and non-numerical contexts provides insight into how learners can develop a deep understanding of mathematical structures and relationships, as demonstrated by Langhorst et al. (2012).

Non-numerical understanding of the part-whole concept involves recognising and reasoning about relationships between quantities without relying on specific numerical values (Langhorst et al., 2012). This early skill is important for young children as they develop mathematical reasoning. Children may understand that a group of apples and oranges together forms a larger whole, without necessarily knowing the exact amounts. Similarly, estimating and comparing groups can also be done intuitively, often without counting. Using an intuitive sense of quantity, children can determine which group has more and which group has less.

Conversely, the numerical understanding of the part-whole relationship involves recognising and working with quantities using numbers, which is crucial for performing arithmetic operations and solving mathematical problems (Langhorst et al., 2012). Children with numerical understanding can recognise and articulate the relationship between the parts and the whole using numbers; for example, if there are five apples (the whole) and two of them are red (a part), then the remaining apples must be three (another part). This understanding is important as it forms the basis of tackling more complex part-whole problems including those that involve initial-unknown scenarios – where the total is known but one part is missing. The foundations of numerical understanding help one to apply their knowledge of numbers to solve such problems systematically. Moreover, numerical understanding encompasses the ability to grasp principles of the four operations. Children learn that these operations can be used to manipulate quantities and that they have specific properties, such as commutativity, that is, $2 + 3 = 3 + 2$.

Both the numerical and non-numerical understanding of the part-whole concept play complementary roles in developing mathematical reasoning and problem-solving abilities (Langhorst et al., 2012). The non-numerical grasp of part-whole relationships serves as a precursor to numerical understanding, allowing children to qualitatively recognise and reason about quantities before expressing these relationships numerically. This progression is essential, as it establishes a strong base upon which more formal arithmetic skills can be constructed. Non-numerical understanding allows children to engage with quantities in real-world contexts and helps them reason through situations without relying on the value of the numbers involved. Non-numerical and numerical understandings of equivalence relations are necessary for a well-rounded comprehension of mathematics that blends intuitive thinking with formal processes. In this study, this can be seen when students manipulate numbers in tasks like $85 - __ = __ - 57$ by providing answers such as 85 and 57, effectively solving the equation without engaging in numerical computation. By recognising that non-numerical understanding allows one to conceptually navigate the relationships between quantities without computation, educators can better support students in transitioning to numerical reasoning. It is essential for educators to help students connect these non-numerical insights to numerical processes, fostering a well-rounded understanding of equivalence and mathematical structures.

Intuitive strategies: Efficiency and sophistication

Research has generally positioned intuitive strategies as a window to developing more sophisticated understanding of concepts in mathematics (see, for example, Boston et al., 2003; Gvozdic & Sander, 2017). Working on problems involving

cross-multiplication, Boston et al. (2003) argue that instead of teaching cross-multiplication as a rote procedure for solving questions on proportionality, learners need to first be given an opportunity to develop intuitive strategies around proportional relationships before making the 'transition from intuitive strategies to cross-multiplication in a way that continues to promote meaning and understanding' (p. 155). This resonates with the argument made by Ricart and Estrada (2022) that learners' intuitive strategies are a necessity in their cognitive development and, therefore, intuitive strategies should be encouraged to give the teacher the opportunity of refining or modifying these strategies in such a way that allows better conceptual understanding. But research is also quick to argue that when intuitive strategies are entrenched, it is usually easy for learners to revert back to these strategies years after being taught other (more efficient) strategies (Gvozdic & Sander, 2018; Ricart & Estrada, 2022). Furthermore, Gvozdic and Sander (2020) argue that 'when the informal strategies are inefficient, teaching students to make way for more efficient ways to find the solution is an important educational issue in mathematics' (p. 111).

Research methods and design

This study took place in the context of a primary mathematics teacher education programme from one university in South Africa. The participants were first-year PSTs, specialising in either Foundation Phase (Grade 1–3) or the Intermediate Phase (Grade 4–6) teaching. The research was part of a design study spanning three cohorts from 2022 to 2024. The participants took a baseline assessment on mathematical equivalence and number relations in February of each year at the commencement of the academic year. The baseline assessment consisted of 27 items, including questions such as missing number task with two unknowns, such as 85 – __ = __ – 57. Follow-up interviews with nine PSTs in line with the patterns of answers informing our conceptual framework and based on frequently appearing answers were conducted. As indicated earlier, in the interviews, we were interested in understanding the reasoning behind the different intuitive strategies provided by the PSTs in the baseline assessment, particularly in the equivalence relation problems involving two unknowns.

Following that, we employed an inductive approach to analyse the data, aiming to generate theoretical insights (Glaser & Strauss, 1967). We began with open coding of the PSTs' various responses. For instance, we observed that some PSTs simply swopped the numbers in the equation, such as interpreting 85 – __ = __ – 57 as 85 – 57 = 85 – 57. In contrast, others demonstrated a range of random and strategic number choices, involving both additive and multiplicative reasoning. This initial exploration led us to conduct follow-up interviews with nine PSTs from the 2022 cohorts using theoretical sampling (Glaser & Strauss, 1967) to gain a deeper understanding of the reasoning behind the PSTs' intuitive strategies. An invitation to a follow-up task-based interview was sent out to all the students who provided correct answers to these two items in the baseline test, and only these nine students accepted and participated in the interviews.

This inductive process generated a set of **six distinct codes** that characterise the PSTs' answers. Subsequently, the interview data enabled us to define and describe each of these codes based on the PSTs' thought processes. We provided justifications for these codes, linking them to relevant literature to support our interpretations and explored hierarchies of sophistication of the solution strategies. A close examination of the six codes through axial coding further categorised them into **three stages** based on comparative relational understanding of equivalence and the kind of mathematical operations involved.

The three stages are: (1) quantitative reasoning, where PSTs simply swopped the two given numbers to make the left-hand side (LHS) the same as the right-hand side (RHS); (2) additive reasoning, where PSTs randomly or strategically choose one unknown, and then worked out the other to make the LHS and the RHS have same value; (3) multiplicative reasoning, where PSTs let both unknowns be x, and solved the resultant equation for x, which leads to multiplicative relations.

In the following section, we first provide an overview of PSTs' general performance in the baseline test on the two open equivalence problems, and then we look deeper into the responses of a sample of participants that they provided during the follow-up task-based interviews.

Overview analysis of pre-service teachers' performance on the open equivalence tasks

Table 1 shows that 54% of the responses were incorrect overall, with a significant improvement in accuracy observed from 2022 to 2024. Incorrect responses declined from 52.8% in 2022 to 46.3% in 2024, reflecting better performance from subsequent cohorts. Also shown in Table 1 are the types of reasoning and strategies used by PSTs in their responses to open equivalence relation questions during task-based interviews, to which we now turn. While an in-depth analysis of incorrect responses is warranted, this is not the focus of the current article, but rather the focus of another article under preparation. The present article has as its specific aim the delineation of PSTs' stages of sophistications and levels of efficiency in their reasoning on equivalence relations questions, as seen in their solutions.

Pre-service teachers' reasoning and intuitive strategies unearthed in the interviews

The findings from analysis of the interview data highlight the various types of intuitive strategies employed, alongside their corresponding stages of reasoning sophistication: quantity-based, additive-based, and multiplicative-based strategies. Additionally, within the additive-based strategies, the analysis revealed four sub-categories, each reflecting different levels of efficiency.

TABLE 1: Summary of pre-service teachers' intuitive strategies across the two tasks for the three cohorts (2022–2024).

Structuring	Reasoning	Strategy	2022		2023		2024		Total	
			Frequency	%	Frequency	%	Frequency	%	Frequency	%
Quantity-based structuring	**Stage1:** Quantitative 156 (14%)	Swopping of given quantities	**70**	16.4	**50**	13.9	**36**	11.5	**156**	14.2
Number-based structuring	**Stage 2:** Additive 188 (17%)	**Level 1:** Large equivalence difference	21	4.9	24	6.7	42	13.4	87	7.9
		Level 2: Using 10 as one of the unknown values	12	2.8	15	4.2	21	6.7	48	4.4
		Level 3: Small equivalence difference	3	0.7	5	1.4	8	2.6	16	1.5
		Level 4: Making the equivalence difference a 10	11	2.6	2	0.6	24	7.7	37	3.4
		Sub-total	**47**	19.0	**46**	12.8	**95**	30.1	**188**	17.1
	Stage 3: Multiplicative 160 (15%)	Repeated variable	**49**	11.4	**76**	21.1	**35**	11.2	**160**	14.5
	Incorrect 598 (54%)	Incorrect or no response	**266**	52.8	**187**	51.9	**145**	46.3	**598**	54.3
-	-	**TOTAL**	**428**	-	**360**	-	**313**	-	**1102**	-

Note: The bold values are the sub-totals in each stage.

Out of the 46% of correct overall responses, 32% involved number-based structuring (making use of both additive and multiplicative reasoning to solve equivalence tasks), while 14% focused on quantity-based structuring (students' ability to swop numbers to enact equivalence).

While mapping the different types of reasoning to the different types of strategies, Table 1 elaborates on the frequencies of occurrence of each of the six strategies seen in PSTs' solutions across the 1102 responses provided by the three cohorts (2022, 2023 and 2024). Within the number-based structuring category, reasoning based on additive and multiplicative principles showed distinct patterns. Additive-based reasoning (Degrande et al., 2018) accounted for 17% of the total, while multiplicative-based reasoning (Matthews et al., 2012) contributed 15%, as can be seen in Table 1. Notably, in the additive-based reasoning category, the strategy involving large equivalence difference (selecting one of the unknown values that makes the equivalence difference big and then determining the other unknown that balances the equation), classified as level 1, recorded the highest frequency of responses at 8%.

When analysing the three cohorts quasi-longitudinally over the three consecutive years, we observed differences in quantity-based reasoning (Stage 1: swopping of given quantities), 16.4% in 2022 and 11.5% in 2024. However, Stage 2, characterised by additive reasoning, was the highest in 2024, at 30.1%. Interestingly, in 2023, Stage 3, which involves multiplicative reasoning, had the highest responses at 21.1%, compared to 11.4% in 2022 and 11.2% in 2024.

Additionally, the use of 10-based strategies within additive reasoning became more common in 2024, where 14.4% of responses employed this approach. The results of 14.4% came about by adding the results of levels 2 and 4 which focus on the use of 10 to find the answer. In other words, students' ability to work with tens and ones informed how they solved these equivalence tasks (Wright et al., 2012). This was a significant rise from 4.8% in 2023 and 5.4% in 2022, indicating a growing trend toward more sophisticated additive reasoning strategies over the three years.

It was through engaging with PSTs' responses and drawing on literature that we developed a framework for categorising PSTs' responses to open equivalence relation questions, to which we now turn.

The emergence of a framework for characterising pre-service teachers' intuitive strategies for solving open equivalence relations problems

Research has shown that learners sometimes solve additive tasks by reasoning multiplicatively and vice versa (Degrande et al., 2018). These insights shed light on a hierarchy in, and development of, PSTs mathematical reasoning. According to Matthews et al. (2012, p. 87), 'the highest level in categorising different types of equivalence tasks is the *Comparative Relational level* (level 4)', which involves recognising the relational definition of the equal sign as the best definition. This relational definition of equivalence allows for viewing these equivalence problems as 'open problems' (Degrande et al., 2018) on two levels, relating to whether the problem is open or closed to being solved: (1) using either quantity-based or number-based structuring, and (2) bringing additive or multiplicative reasoning to bear (see Figure 1).

To reiterate, equivalence relations problems are either closed or open. In open equivalence relations problems, additive and multiplicative reasoning are brought to bear during solving. Closed equivalence relations problems are amenable only to additive reasoning. For instance, 85 – 75 = _ – 57 is a closed equivalence relations problem because the only way to arrive at the answer of 67 is additively, regardless of whether the answer is generated following elementary counting-on processing or an advanced algebraic procedure. On the other hand, 85 – _ = _ – 57 is an open equivalence relations problem, firstly at the level of structuring, because

it can be solved using quantitative (non-numerical) structuring – that is, without the need for a full computation procedure – by simply returning the given quantities in the order in which they were given (as in 85 – 85 = 57 – 57) or in the reverse order (85 – 57 = 85 – 57), or it can be solved numerically with a formal procedure that involves number-based structuring. While quantity-based structuring reflects pure quantitative (non-numerical) reasoning, number-based structuring can reflect either additive or multiplicative reasoning. Thus, we have used the idea of an 'open word problem' by Degrande et al. (2018) and adapted it to two unknowns, equivalence problems of the form a – _ = _ – d which we have designated 'open equivalence relations problems'. Importantly, as represented in Figure 2, given that the problem has two unknowns, instead of a unique solution, there is an entire range of pairs of solutions that will apply.

We figured, as seen in Figure 2, that if the task 85 – _ = _ – 57 were to be solved algebraically, one could find the answer by picking a value for the unknown on the LHS of the equation, say 10, letting the other unknown be x, and then algebraically solving for x: $x = 85 – 20 + 57$, $x = 65 + 57$, so $x = 122$. Given that the algebraic solution only involved additive operations, it would be appropriate to designate the reasoning brought to bear in this case as additive reasoning. Whilst this was the sense we had in our initial analysis based purely on the answers provided by the PSTs, this is precisely what we found to be true during the interviews when we probed PSTs for their solutions.

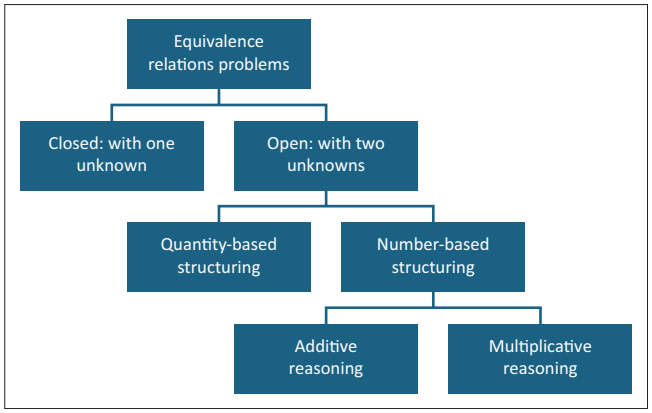


FIGURE 1: A view of open equivalence relations problems revealing the structuring and reasoning underpinning their solutions.

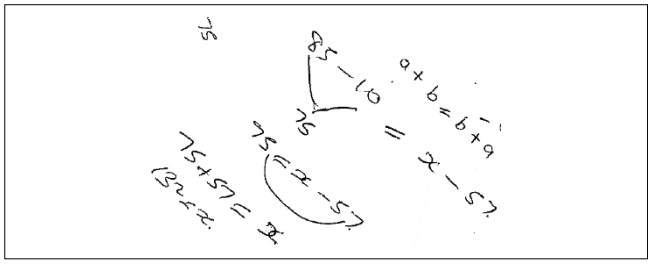
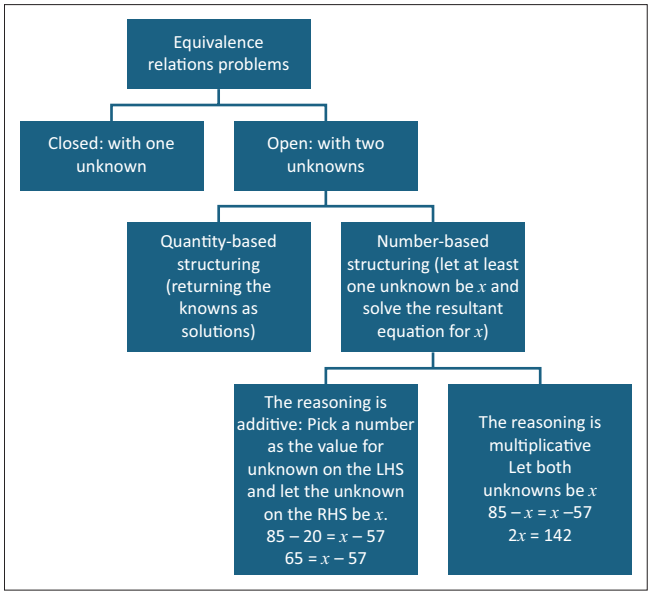


FIGURE 3: Example of additive strategy extracted from pre-service teachers' task-based interview work.

As confirmed in Table 1, the level 2 strategy predominantly used by PSTs was the one (captured in Figure 3) wherein PSTs typically chose 10 as one of the unknowns, and then worked out the equivalence difference to be 75, and then let the other unknown be x and worked out the value of x that will solve the equation.

Alternatively, one could let both unknowns be x, and solve the resultant equation for x: $85 – x = x – 57$. In such this case, solving for x would lead through the steps $2x = 85 + 57$, $x = 142/2$ to arrive at the answer of 71. Given that the algebraic solution amounts to calculating an average, which invariably involves division by 2, it would be appropriate to designate the reasoning brought to bear in this case as multiplicative reasoning. As a result, two-solution equivalence relations problems of the form a – _ = _ – d are open to being solved additively or multiplicatively. The two problems used in this study (85 – __ = __ – 57 and 97 – __ = __ – 69) were carefully chosen in such a way that the average is a whole number, which is often the type of number that would be expected for the answers.

Research emphasises the importance of incorporating algebraic approaches alongside arithmetic strategies when it comes to solving equivalence tasks. According to Alibali et al. (2007), learners need to transition from relying solely on arithmetic processes to integrating algebraic methods to navigate higher-level tasks. This transition involves reframing the equal sign from denoting an 'operation – equals



RHS, right-hand side; LHS, left-hand side.

FIGURE 2: A view of equivalence relations problems revealing the structuring and reasoning underpinning the algebraic processes used to solve them.

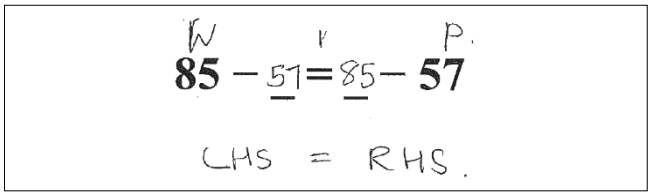


FIGURE 4: Example of quantity-based structuring extracted from pre-service teachers' task-based interview work.

– answer' structure to representing 'same as' (Matthews et al., 2012, p. 87). For example, while in primary school learners would be encouraged to focus on the difference between the numbers on the LHS of the equation 85 – 75 = _ – 57 in order to determine the missing number on the RHS of the equation that would maintain the equivalence, learners in secondary school would be confronted with a variable (85 – 75 = x – 57) in the place of the unknown, and would be expected to use an algebraic procedure to find the missing number (67).

Characterising stages of sophistication for solving open equivalence relations problems

Quantitative reasoning

We have designated quantitative reasoning a demonstration of a non-numerical quantitative understanding of the part-whole concept in the context of solving equivalence relations problems. The strategy involves simply swopping the given numbers to make the LHS and the RHS the same (see Figure 4). While this method is the quickest way to obtain answers, it often yields a localised or singular solution.

As summarised in Figure 2, the data revealed that PSTs reasoning can be broadly classified as either additive or multiplicative. Likewise, their structuring can be broadly classified as either quantity-based or number-based.

Additive reasoning

As indicated in Table 1, within the additive reasoning band associated with numerical structuring, four different intuitive strategies emerged from the data set. This is where PSTs randomly or strategically choose the value of one of the unknowns, and then work out the other unknown additively.

Table 2 summarises the four different intuitive strategies that we classify as demonstrating additive reasoning. From this level, we notice that the initiative strategies are generalisable. In further scrutinising the strategies within the additive reasoning, we noted a hierarchy of efficiency in getting to the answer. This understanding is driven by literature on efficiency with base-ten thinking (Wright et al., 2012), and strategic competency (Askew et al., 2019; Kilpatrick et al., 2001) in thinking around any mathematical problems.

TABLE 2: Levels of intuitive additive-based reasoning.

Levels of efficiency	Strategy	Description
Level 1	Large equivalence difference	Selecting one of the unknown values that makes the equivalence difference bigger and then determining the other unknown that balances the equation. For example, 85 – **13** = **129** – 57
Level 2	Using 10 as one of the unknown values	Selecting 10 to be one of the unknowns, and then determining the other unknown that balances the equation. For example, 85 – **10** = **132** – 57
Level 3	Small equivalence difference	Selecting one of the unknown values that makes the equivalence difference smaller and then determining the other unknown that balances the equation. For example, 85 – **83** = **59** – 57
Level 4	Making the equivalence difference a 10	Selecting the value for the unknown to give equivalence difference of a 10 on both sides. For example, 85 – **75** = **67** – 57

Note: The bold numbers are where the unknowns would have been, representing the possible answers in each level.

Overall, the efficiency levels highlight a progression from less efficient, trial-and-error approaches (level 1) to more strategic and refined methods (level 4).

Multiplicative reasoning

We also noted that some PSTs used multiplicative relations in working out the two unknowns. The interview data revealed that these PSTs use one variable to represent the two unknowns. For example, 85 – __ = __ – 57 is interpreted as 85 – x = x – 57 and hence they now work out the value of x. The two unknowns have the same value (see the excerpt in Figure 5).

Despite the additive structure of the equation (given the minus signs separating the known and the unknown on either side of the equal sign or equation), the computation of the solutions implies multiplication and division as mathematical operations. Thus, the algebra is multiplicative as the solution is the average value of the two given numbers. In the task 85 – _ = _ – 57, the solution was to add the two numbers together (85 + 57), and divide the resulting 142 by 2, to arrive at the answer of 71 (see Figure 5a). Similarly, in the other task (97 – _ = _ – 69), the solution was to add the two numbers together (97 + 69), and divide the resulting 166 by 2, to arrive at the answer of 83 (see Figure 5b). We have designated this a demonstration of a numerical multiplicative understanding of the part-whole concept in the context of solving open equivalence relations problems. We called, this '*repeated variable*' strategy because it requires assigning the same variable to the two unknowns.

Hierarchies within the categories of pre-service teachers' reasoning

The discussion so far has focused attention on description of the range of PSTs' intuitive strategies and the categories that constitute their ways of reasoning as either numerical or non-numerical. This discussion is indeed necessary and structurally important to our understanding of the PSTs' strategies for solving open equivalence problems. However, a further crucial feature relates to whether any hierarchy exists across the categories that describe different ways that reasoning moves to more sophisticated ways of thinking. Our thought about hierarchies across categories was driven by the comparative relational nature of the strategies, and the need for understanding progression in the design of instruction for teaching equivalence problems as a precursor to early algebraic thinking.

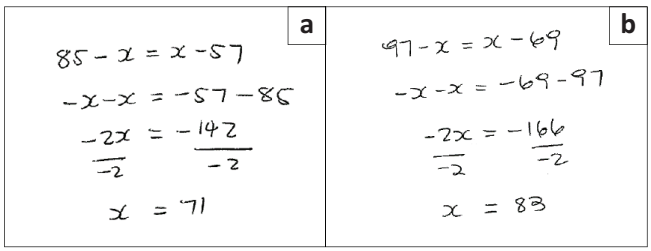


FIGURE 5: Examples of multiplicative-based strategy extracted from pre-service teachers working during one of the interview sessions.

TABLE 3: Stages of intuitive reasoning.

Stage	Reasoning	Description
Stage 1	Quantitative reasoning	This is a strategy where PSTs simply swopped the two given numbers to make the LHS the same as the RHS
Stage 2	Additive reasoning	PSTs randomly or strategically choose a value for one unknown, and then work out the other unknown to make the LHS equal to the RHS
Stage 3	Multiplicative reasoning	PSTs use a variable to represent both unknowns, and algebraically solve the resultant equation

RHS, right-hand side; LHS, left-hand side; PSTs, pre-service teachers.

The literature base relating to sophistication suggests the hierarchical nature of the three categories, with multiplicative reasoning demonstrating a high sophistication level, and quantitative (non-numerical) the least level of sophistication (Degrande et al., 2018; Langhorst et al., 2012; Matthews et al., 2011). Based on the relevant literature, we concluded with a three-stage hierarchy of sophistication across the different categories of reasoning, as reflected in Table 3.

The three stages of reasoning highlight the progression of PSTs' conceptual development, moving from surface-level singular problem-solving solution to more complex, abstract thinking. The implications for teaching are clear: scaffolding PSTs' growth from concrete to abstract reasoning through targeted, stage-appropriate interventions is essential for fostering deeper mathematical understanding.

In the following section, we present the implications of findings from our study, emphasising how they can inform instructional approaches and curriculum development to better support PSTs' conceptual growth.

Implications of the findings

In this study, we have defined open equivalence relations as equivalence problems which, by virtue of having two unknowns, are open to being solved additively or multiplicatively. By introducing these problems as belonging to the *Comparative Relational* category of equivalence tasks, this study has elaborated on the highest category of equivalence tasks in Matthews et al.'s (2012) categorisation of different types of equivalence tasks. Also, this study offers a possible analytical framework for analysing levels of efficiency in human (educators, students, learners) reasoning when solving open equivalence relations problems. Consequently, the findings have implications for teacher training programmes and intervention studies with an interest in building sophistication and efficiency into PSTs' strategies for solving open equivalence relations problems.

Implications for curriculum development

Both the numerical and non-numerical understanding of the part-whole concept play complementary roles in developing mathematical reasoning and problem-solving abilities (Langhorst et al., 2012). The non-numerical grasp of part-whole relationships serves as a precursor to numerical understanding, allowing children to qualitatively recognise and reason about quantities before expressing these relationships numerically. This progression is essential, as it establishes a strong base upon which more formal arithmetic skills can be constructed. Non-numerical understanding allows children to engage with quantities in real-world contexts and helps them reason through situations without relying on the value of the numbers involved. Non-numerical and numerical understandings of equivalence relations are necessary for a well-rounded curriculum that blends intuitive thinking with formal processes.

In this study, this can be seen when students manipulate numbers in tasks like 85 – __ = __ – 57 by providing answers such as 85 and 57, effectively solving the equation without engaging in numerical computation. By recognising that non-numerical understanding allows one to conceptually navigate the relationships between quantities without computation, educators can better support students in transitioning to numerical reasoning. It is essential that educators help students connect these non-numerical insights to numerical processes towards fostering a well-rounded understanding of equivalence and mathematical structures.

Recognising that the differences on either side of the equal sign represent the parts that must be equivalent encapsulates how part-whole relationships are foundational to both numerical and non-numerical understanding in mathematics. Thus, it is important the curriculum to be designed so that prospective teachers to make the connection between non-numerical and numerical understanding of the part-whole structure of all equivalence relationships.

Implications for future research into equivalence relations

This study offers a possible analytical framework for analysing levels of efficiency in students' reasoning when solving open equivalence relations problems. The framework highlights the existence of a progression in the solving of open equivalence relations problems, one that begins with a quantity-based non-numerical swopping action to maintain equivalence to a multiplicative reasoning-informed number-based structuring to finding the correct answer.

From a structuring point of view, when PSTs swopped numbers to find the answer, they were demonstrating quantity-based structuring. At this level of structuring work, PSTs have a basic notion that both sides of the equal sign need to be a similitude one of the other in order for the balance to be maintained, hence the swooping of the numbers. Serendipitously, this similitude of the left-hand side with the right-hand side of the equation presents a convenient place to start in the learning and teaching of open equivalence relations problems, and learners' development from non-numeric reasoning towards algebraic thinking. In this progression, the intermediate phase is characterized by number-based structuring that relies on additive reasoning. In this study, while most of the responses were associated with additive reasoning, there were those PSTs who could apply multiplicative reasoning to produce the answer. The implication of the analysis points researchers to the possibility of reordering this cross-sectional view of PSTs' reasoning on, and structuring of, their solutions to open equivalence tasks

into a longitudinal view of the development of their and their future learners' understanding of these types of problems.

This study corroborates what previous research has shown: that a deep understanding of mathematics is a function of an equally deep understanding of structure (see for example Mulligan, 2002; Tondorf & Prediger, 2022). And as Mason et al. (2009), and Mulligan and Mitchelmore (2009) would argue, structure permeates the whole of mathematics at every stage and for every age. In the case of our study, PSTs need to realise that these structures exist when solving open equivalence questions, and that when selecting the pairs of solutions, both the complexity and the sophistication of getting the answers will be impacted. In other words, when learners see the structure as quantity-based structuring, they will operate in a less complex way together with less sophisticated strategies in producing the answers. PSTs who use number-based structures to solve equivalence tasks demonstrate a high ability to work with the complexity of mathematics and sophistication of strategy. Working with more complex ways of solving equivalence tasks and with higher sophistication of strategy develops algebraic thinking. In other words, the framework provides researchers and practitioners with a means to a deep understanding of open equivalence relations problems which includes knowledge of learners' placement in the progression from quantity-based manipulations to reasoning multiplicatively about the relations in a number-based calculational approach.

Implications for teaching and learning

Ultimately, it is this 'open stance' that is the quintessential specialised knowledge to be honed for teaching: the ability to solve open equivalence relations problems in one of two qualitatively different ways – additively and multiplicatively – and to provide one or the other solution appropriate to a learner's level of understanding. Put simply, when it comes to equivalence tasks, teachers who can see that an open equivalence relations problem lends itself to being solved both additively and multiplicatively and possess the ability to solve and explain it in both ways will be best placed to support their learners across the attainment range. Exposing learners to both the additive and multiplicative approaches to solving these equivalence tasks promotes flexibility in the use of different strategies and opens the way toward deepening learners' understanding of equivalence.

Conclusion

At the outset of our study, we proposed several hypotheses, including the idea that the development of level 4 ability of Donovan et al.'s (2022) Construct Map, that is, solving equations and analysing their structure by comparing both sides of the equal sign (Rittle-Johnson et al., 2011, p. 88), begins in elementary school, and that these equivalence relations problems should pose no major challenges to pre-service mathematics teachers. It was therefore surprising to find that more than half of the first-year PSTs who are preparing to teach mathematics in primary schools cannot correctly answer equivalence relations problems with two unknowns. We consider this finding a call to action. The work that needed to be done in fulfilment of this call to action should begin with delving deeper into the incorrect answers to interrogate the reasoning behind them.

But beyond the issue of incorrect answers, the findings from our study underscore the importance of identifying a suitable starting point for gradually introducing PSTs to various strategies for solving equivalence relations problems. At the same time, it opens a space to begin a conversation with PSTs regarding the need for building flexibility and sophistication into their use of strategies for equivalence relation problems. More generally, this finding underscores the need for more pinpointed attention to be paid to developing teachers' knowledge of equivalence relations problems and how to solve them. For PST education programmes, therefore, exposing PSTs to the different levels of sophistication, and the different stages of reasoning, as well as knowledge of the different structures associated with solving equivalence tasks, will be key considerations.

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Competing interests

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Authors' contributions

H.M.T. contributed to the initial conceptualisation, data coding and analysis, writing the first draft of the manuscript, and overall coherence of the article. L.A. contributed to the writing of the methodology, formal analysis and the writing and reworking of the final manuscript. S.H. contributed to the data collection and analysis, as well as writing and reworking parts of the manuscript. C.M. contributed to the data collection, analysis, writing and reworking of parts of the manuscript. A.A.E. contributed to the initial conceptualisation and editing of the final manuscript, and the overall coherence of the article.

Ethical considerations

Ethical clearance to conduct this study was obtained from the University of the Witwatersrand (Wits), Human Research Ethics Committee (No. H23/07/10).

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Data availability

The authors declare that all data that support this research article and findings are available in the article and its references.

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