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Problem-solving in geometry: A multilingual perspective through commognitive and Polya's steps of problem-solving

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Based on the first author's doctoral research, this article explores the connection between Sfard's commognition theory and Polya's four steps of problem-solving. The article highlights how commognition, through its focus on language and discourse, helps explain learners' problem-solving approaches as either ritualistic or explorative. The study emphasises that for multilingual learners, understanding mathematical problems, especially in geometry, depends on grasping terminology and visual elements. By combining commognition with Polya's framework, the analysis suggests that learners' problem-solving and visualisation skills can be improved, though language remains a key challenge.

Contribution: This article establishes a theoretical connection between Sfard's commognition and Polya's problem-solving steps, highlighting their complementary roles in mathematics education. The article validates how commognitive elements, particularly word use and visual mediators, support multilingual learners in understanding and solving geometry problems. The study offers insights into how discourse influences learners' problem-solving routines, with implications for teaching in linguistically diverse classrooms.

Keywords: commognition; Polya's steps of problem-solving; geometry problem-solving; multilingual learners; language and discourse; visualisation.

Introduction

In the South African context, the underachievement of learners in mathematics has been identified as a key concern (Geary, 2011; Mabena et al., 2021; Reddy et al., 2020). In addition, the 2023 National Senior Certificate (NSC) Diagnostic Report identified specific areas of difficulty in Euclidean geometry, underscoring the need for targeted interventions to enhance learners' comprehension and performance in this subject area. Euclidean geometry is a branch of mathematics that deals with points, shapes, spatial figures, and space (Bassarear, 2012; Biber et al., 2013; Luneta, 2015). It contributes to logical reasoning, visual-spatial intelligence, and abstract thinking (Kösa, 2016). Problem-solving plays a significant role in mathematics education, enabling learners to practise and integrate the concepts, theorems, and skills that they learn (Caerols-Palma & Vogt-Geisse, 2022; Hudoj, 2005; Tambaram, 2019). For multilingual learners who may struggle with language-based explanations, visualisation offers an alternative pathway to understanding complex geometric concepts, thereby enhancing their ability to solve geometry problems. However, multilingualism has also been identified as one of the main reasons learners experience challenges in understanding mathematics. According to Barwell (ed. 2009) and Sharma and Sharma (2023), learning mathematics in multilingual classrooms is complex because it requires teachers to support learners with diverse educational needs, each of which, when considered individually, could require distinct interventions. Expanding on Barwell's notion, Sharma and Sharma add that learning mathematics in a language that is neither the learners' first language nor the teacher's can create challenges such as code-switching, mediation, and a lack of transparency in communication. Given these complexities, Essien (2013) emphasises that it is crucial for teachers to equip learners with the necessary understanding and skills to navigate the challenges posed by learning mathematics in multilingual classrooms. A growing body of research highlights that, in South Africa, learners face difficulties in understanding geometry (Tachie, 2020). For instance, a study by Mudhefi et al. (2024) found that Grade 12 learners demonstrated poor conceptualisation of geometric properties and terminology, leading to difficulties in understanding and applying geometry concepts. Additionally, research by Naicker (2021) highlights that learners' lack of engagement in meaningful learning situations, compounded by language barriers and inadequate teaching methodologies, contributed to their

difficulties with geometry. Essien argues that learners in multilingual contexts must be adequately prepared to manage the complexities of learning mathematics in a language that is not their native tongue. In addition, Sfard (2008) argues that, in order to equip learners to deal with the difficulties they face in mathematics, the focus must shift to the transformation of discourse participation; hence, she thought that learner understanding of mathematical thinking is based on how they communicate about mathematics. Sfard theory has been applied in other mathematical domains in South African research, such as calculus (Siyepu & Ralarala, 2014), algebra (Roberts & Le Roux, 2019), functions (Mpofu & Mudaly, 2020), the use of technology as a visualisation tool (Zulu & Mudaly, 2023), geometry (Mahlaba, 2021), and teachers' discourse (Berger et al., 2013; Berger & Bowie, 2012; Van Jaarsveld, 2018), numeracy (Heyd-Metzuyanim & Graven, 2016), equations (Roberts & Le Roux, 2019) and geometry (Hlongwana et al., 2025). The importance of commognition has been recognised for its potential to address challenges across the domain of mathematics. Polya's steps of problem-solving are specifically dedicated to guiding multilingual learners in understanding the challenges they encounter during problem-solving for this study in geometry. Although these theories are not located in one domain of mathematics, in this article, we argue that they are currently serving the same purpose: improving learners' understanding of mathematics problem-solving. The primary objective here is to outline the vital role of commognition in enhancing learners' problem-solving understanding, using Polya's four steps. As South African multilingual learners face a serious challenge in comprehending both the language of instruction and mathematical vocabulary simultaneously (Mukuka & Alex, 2024), we thus discuss the elements of each theory, then debate how the study incorporated commognition and Polya's four steps to enhance the understanding of geometry problem-solving questions. The findings reported in this theoretical article are rooted in a bigger doctoral study. Still, this article focuses only on the relationship between commognition and Polya's four problem-solving steps in solving geometry problems in a multilingual setting in Grades 10–12. Thus, no empirical data will be cited in this article. While both Sfard's commognition theory and Polya's problem-solving framework have been widely discussed in mathematics education, there remains a noticeable gap in research that examines how these two theories can be integrated to support learners, specifically in Euclidean geometry, for learners who learn mathematics using a language that is not their native tongue. Hence, the reintroduction of geometry into mainstream (i.e, Mathematics P2) adds to the already crumbling pass rate in mathematics (Govender & Amevor, 2025). This gap is especially significant in multilingual contexts such as South Africa, where language barriers often hinder learners' ability to engage meaningfully with mathematical content. Furthermore, according to Barwell and Sharma and Sharma, learning mathematics in multilingual classrooms is complex because it requires teachers to support learners with diverse educational needs, each of which could, in turn, require distinct interventions. However, most existing studies treat discourse and problem-solving as separate domains, with limited attention paid to how a combined theoretical model might support a deeper understanding and strategic thinking in geometry. Furthermore, there is a shortage of frameworks explicitly designed to address the unique challenges multilingual learners face when solving geometry problems. This article aims to close these gaps by exploring how integrating commognition and Polya's four steps can offer a cohesive, language-responsive approach to teaching geometry problem-solving.

Literature review

Multilingualism in mathematics education

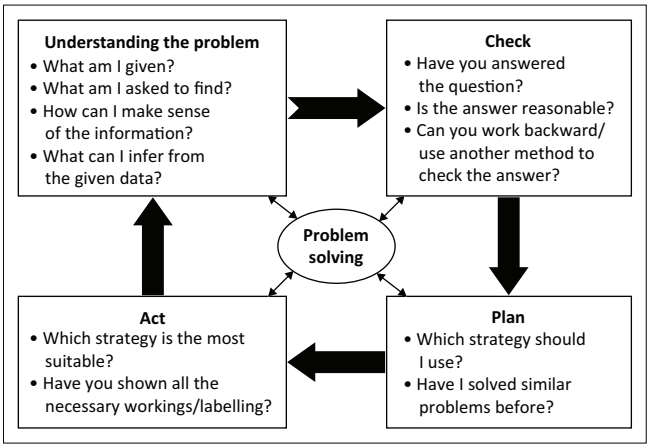
Multilingualism has been defined in various ways in the literature. Kemp (2009) notes that monolinguals are speakers of one language, while bilinguals are speakers of two languages. Multilinguals generally use or know three or more languages, with varying degrees of proficiency in these languages (Barwell, 2018; Bose & Clarkson, 2016). However, scholarship in the field of multilingual research in the last decade has expanded the definition of multilingualism to include 'various forms of social, institutional and individual usage as well as individual and group competences, plus various contexts of contact and involvement with more than one language' (Franceschini, 2009, p. 29). This understanding of multilingualism acknowledges language diversity, includes sensitivity towards socio-cultural diversity, and appreciates society's heterogeneity. It is now used as an umbrella term that encompasses research on bilingualism. For this article, a contextual understanding of multilingualism is adopted. Language plays a critical role in teaching, learning, and mathematical problem-solving (Barwell, 2014; Morgan et al., 2015; Moschkovich, 2017; Planas, 2018). Having knowledge of more than one language has been recognised as a 'resource' for learning mathematics (Adler & Ronda, 2015; Moschkovich, 2013, 2015), a recognition that was not the case in earlier years of multilingual studies in mathematics education research. Research on multilingualism has significantly influenced the field of mathematics education. Research on multilingual education has moved beyond deficit theories of multilingualism to address how the linguistic resources multilingual learners bring to class can be effectively harnessed to provide high-quality mathematics education. In other words, more research now focuses on multilingualism as a resource in multilingual mathematics classrooms (Barwell, 2018; Planas, 2018; Prediger, 2019; Ryan & Parra, 2019). However, many researchers highlight the complexity of learning and teaching in multilingual classrooms, where learners are simultaneously learning the language of learning and teaching (LoLT) and learning mathematics as both a discipline and a language. In multilingual settings, learners who share the LoLT with their home language are more familiar with the linguistic structures encountered in mathematics classrooms (Barwell, 2018). However, research indicates that this is not the case for learners whose home language differs from the LoLT (Gohel et al., 2007; Robertson & Graven, 2018). When learners' home

language differs from the LoLT, they must contend with the additional challenge of not being fluent in the language of instruction. Mathematics education involves problem-solving, which requires a firm grasp of mathematics vocabulary. This suggests that learners must develop proficiency in the language of instruction to comprehend mathematical terminology during problem-solving effectively. The LoLT is thus expected to play a crucial role in learners' understanding of mathematical concepts. In multilingual contexts, research has demonstrated that while multiple language use in the mathematics classroom can support learning (Moschkovich, 2015), it is not a straightforward solution, as diverse language infrastructures within and around schools place varying demands on learners and their teachers. While a review of the literature on multilingual learners and mathematics highlights research conducted in various countries, there is a limited number of studies on this topic that specifically focus on the integration of Sfard's (2008) commognition tenets and Polya's (1945) four steps of problem-solving on how they complement each other in supporting multilingual learners' understanding during problem-solving in geometry (e.g. Barwell, 2014, 2018, 2020; Moschkovich, 2002, 2007, 2012, 2018, 2019; Takeuchi, 2015, 2016). While many studies examine the socio-political dimension of language as a resource for gaining access to mathematical knowledge, they do not provide insights into how understanding of mathematical concepts is enhanced during geometry problem-solving (Planas & Setati-Phakeng, 2014; Setati & Barwell, 2006). The emphasis on language in these studies often overshadows learners' understanding of mathematical concepts and the procedural framework that guides learners through geometry mathematical problem-solving, thus shifting the central aim of mathematics education research to the periphery. Therefore, the objective of this study is to build on existing research by providing an in-depth analysis of the commognition framework and Polya's four steps of problem-solving during problem-solving for learners who study mathematics in a language that is not their native tongue. Their connection serves the different objectives; however, in this article, we argue that they can assist in analysing learners' understanding of geometry concepts when solving geometry questions in multilingual settings.

Polya's problem-solving steps

According to Hanegem (2017), mathematics education is about encouraging mathematical thinking and understanding, and developing problem-solving abilities and research skills. Many mathematical problems can be solved using several different procedures (Gcasamba, 2014). Polya's (1945) four steps of problem-solving are not designed to restrict learners to using a particular strategy to solve a task but, instead, provide a guide for understanding the problem and choosing an approach based on the learner's understanding of the problem.

Polya's four-step problem-solving process is shown in Figure 1.



Source: Adapted from Caglayan, G. (2014). Static versus dynamic disposition: The role of GeoGebra in representing polynomial rational inequalities and exponential-logarithmic functions. *Computers in the Schools*, 31(4), 339–370. https://doi.org/10.1080/07380569.2014.967632

FIGURE 1: Polya's four-step problem-solving process.

Understanding the problem

The first step of Polya's (1945) four-step problem-solving model is understanding the problem, which involves carefully reading the problem statement, listening attentively, and identifying key terms embedded in the problem. This step is crucial because it determines whether a learner has fully grasped the problem. In mathematical problem-solving, learners must understand the concepts involved, establish connections through visual perception, and enhance their critical thinking skills. At this stage, learners should visualise the problem with a strong conceptual understanding to develop an appropriate strategy. For instance, in Grade 11, if it is stated in geometry that line TP is a tangent to a circle passing through ACD, multilingual learners need to understand these concepts to move on to the next step: devising an appropriate algorithm to solve the problem.

Plan

The second step, devising a plan, involves learners formulating a strategy or setting up an equation to solve the problem. At this stage, learners must determine an appropriate approach before attempting a solution. They may also draw on prior problem-solving experiences and integrate their existing knowledge into their approach to the new problem. For multilingual learners, this step may involve selecting a strategy based on their comprehension of the problem statement, the accompanying geometry diagram, or their prior knowledge related to the question.

Act

The third step, carrying out the plan, requires learners to implement their chosen strategy. If they have formulated an equation or identified a relevant approach, this is the stage where they execute their solution process. Here, participants engage in problem-solving using the strategy they deem most effective. At this stage, multilingual learners are expected to use geometry theorems, axioms, and other geometric properties to solve any problems. The use of the above properties

requires multilingual learners to understand both the language of teaching and learning and mathematical language. Language plays a crucial role in teaching, learning, and mathematical problem-solving (Moschkovich, 2017).

Check

The fourth step, looking back, involves reviewing the solution to determine whether all given data was used appropriately and whether the answer is reasonable. At this stage, learners verify their solutions in accordance with mathematical principles, such as the correct use of geometry properties, like verifying that the adjacent angles on a straight line add up to 180°. Furthermore, multilingual learners assess their solutions using geometry shapes, properties, and theorems to determine whether their answers make sense or if alternative methods should be considered, for example in verifying that the sum of opposite angles of a cyclic quadrilateral adds up to 180°. Polya's four-step problem-solving process is particularly relevant to studying how multilingual learners approach geometry problems. By observing learners' responses from step one to step four, the researcher can assess their comprehension of the situation before solving it. This, in turn, enables an analysis of how visualisation influences problem-solving for multilingual learners in geometry.

Commognition

This section provides a detailed overview of Sfard's (2008) theory of commognition, with a specific focus on the theory that relates to this article. The section commences with an overview of the theory, its key elements, and its relevance to analysing multilingual learners' thinking when solving geometry problems.

Overview of commognition theory

Commognition theory originates with the idea that thinking is a form of communication with oneself. The term commognition is derived from 'communication' and 'cognition', and refers to the relationship between thinking and communication. Sfard (2008) asserts that thinking is an individualised form of communication, whereas communication is 'a collectively performed rule-driven activity' that directs the activities of groups or individuals (Sfard, 2008, p. 118). Commognition theory offers a discursive framework for studying and interpreting an individual's activity to gain insights into the 'intricacies of learning' (p. 566). Sfard asserts that learning is an individual development in a 'patterned collective activity' (p. 570), often guided by teachers. In commognitive theory, communication may be expressed in the form of text, spoken, artefacts, and the use of tangible objects. Sfard asserts that 'different types of communication set apart by their objects, the kinds of mediators used, and the rules followed' (p. 93) that bring participants together in a community of communication, are called discourses. Discourses are, therefore, tools of communication, such as keywords (spoken and textual) and their uses, as well as visual mediating tools used to communicate concepts to others in regulated activities – in this article, during problem-solving. This outlines a link between discourse and communication in commognitive theory. Problem-solving can also be considered a form of discourse that follows patterned activities through which concepts are communicated in learners' minds when solving mathematical problems (Zulu & Mudaly, 2023). Problem-solving engages learners in a purposive discourse to enhance mathematical critical thinking (Carpenter & Clayton 2014). In commognition theory, thinking is linked with learning. Thinking, as a personalised form of communication, when associated with school learning, deals with the process of restructuring and extending one's discourse on an object of study with the assistance of teachers (Ben-Zvi & Sfard, 2007; Sfard, 2008). Communication is the mediating tool between teachers and learners in classroom discourse, making teaching a social interaction through which learners engage in purposive, organised activities, such as problem-solving, to construct knowledge. Learning occurs when learners modify their existing knowledge or develop new knowledge through current activities (Sfard, 2008). In this article, while solving geometry problems, multilingual learners' knowledge may be modified based on their understanding of geometric shapes, concepts, theorems, and axioms, and on their ability to comprehend geometry problem statements through their mathematical discourse during cognitive processing. However, the language of instruction might play a vital role in multilingual learners' understanding of the above properties, and this understanding might be challenged when they experience cognitive load in processing both geometry vocabulary and the language of instruction. This article seeks to identify ways to bridge language barriers for multilingual learners, enhance their understanding, and improve their problem-solving skills in geometry. According to Robertson and Graven (2019), language can either support or hinder certain groups of learners in making sense of mathematical concepts in geometry.

Mathematics as a discourse

Human beings can play a role in different types of discourse, but may not be equipped to participate in all discourses. For instance, it might be difficult for visual artists to participate in a specialised mathematical discourse, just as it would be difficult for a mathematician to join in a discourse about project management or history. Sfard (2008) maintains that mathematics is a type of special discourse that is characterised by many forms of communication tools, which are known to those engaged in the discourse. Mathematics discourse comprises the communication of ideas such as concepts, proofs, laws, and theorems that are generally limited to mathematics. This shows that mathematics discourse is characterised by communication tools that are recognised by fellows of a unified community. Sfard describes mathematics as 'an autopoietic system' (p. 161), in which its end results are precisely the constituents of its present discourse, as its objects. For the purposes of this article, this view emphasises that, in geometry, diagrams, mental imagery, and spatial reasoning are integral to mathematical discourse.

Commognition further describes mathematical learning as participation in mathematical discourse (Sfard, 2008). The significance of communication in learning is highlighted by Vygotsky (1978), and Lave and Wenger (1991) highlight the importance of participation in the process of learning. In addition, the idea of mathematics as a discourse has been strengthened by researchers such as Nardi et al. (2014) and Sfard (2014), who argue that not considering mathematics as a discourse would be ridiculous. The objects produced in mathematical discourse are described as abstract discursive objects comprising mathematical magnifiers. They are the products of objectification (Sfard, 2008), meaning that they exist as shared cultural objects that arise from collective discourse, and their meaning is shaped by the interactions and communication of individuals. In other words, mathematical objects are not simply 'discovered' in an objective sense but are created and understood through conversations, discussions, and mathematical activities such as problem-solving questions. Sfard (2008) distinguishes between two types of discourse: colloquial and mathematical discourse. Colloquial discourse is everyday-life discourse, visually mediated by pre-existing objects (Sfard, 2008). Colloquial discourse of mathematics utilises everyday language that is reified, and colloquial narratives can be endorsed by multilingual learners through their engagement in discourse with the knowledgeable other or through repetition. This view refers to how multilingual learners can reinforce their understanding and problem-solving skills in geometry by engaging in conversations with someone who has more knowledge, such as a teacher, peer, or mentor. These interactions help learners to process and internalise concepts in their own words, making the learning more meaningful. What makes mathematical discourse different is that it is specific to mathematics. Sfard (2008) states that mathematical discourse can be identified by four elements: word usage, visual mediators, routines, and narratives. These are discussed next.

Word usage

According to Sfard (2008), a discourse is characterised by the types of keywords it uses. This refers to the vocabularies or terminologies of communication. The kind of words used in a discourse gives it a unique character. In mathematics discourse, the words, terms, or languages used must have a specific mathematical meaning. For example, the use of words such as 'tangent,' 'diameter,' 'radius,' 'alternating angles', and 'perimeter' indicates that the communication is a mathematical discourse (Nardi et al., 2014; Remillard & Kim, 2017; Sfard, 2008). Some words have a more general meaning in everyday discourse but have a narrow or specific meaning in mathematical discourse. For example, the word 'half', when used in the sentence 'I ate half of a loaf of bread' in everyday discourse, could mean 'approximately half', but the same word in mathematical discourse means exactly half (Berger, 2013). Similarly, the word 'angle' in everyday discourse might imply a corner or a divergence in opinions, but in mathematics (geometry), it is a figure formed by two rays, called 'sides of the angle', sharing a common ending point called the 'vertex'. While a word used in literate mathematical discourse has an exact and definite meaning in mathematics, this is not the same in colloquial discourse: colloquial discourse emerges spontaneously from daily communication or talk and may have different meanings to different people. Having the understanding of the word used in mathematical discourse, the study explored the contributory factor to poor performance in geometry for learners whose language of LoLT is not their native language, using the geometry problem statement alone. Second-language learners require scaffolding to progress in both linguistic and mathematical skills (Moschkovich, 2012). Hence, geometry questions comprise the vocabulary used in problem statements, such as tangent, chord, radius, and diameter. This can help identify the most contributory factor to learners' poor performance in geometry in multilingual contexts. In a mathematics classroom, learners make sense of mathematical objects when the words of the discourse have an ordinary meaning in all mathematical discourse (Sfard, 2007). Similarly, in mathematics problem-solving questions, learners make sense of the question when the words used in the discourse have an ordinary meaning to all members of the mathematical community. Mathematical words that are used to signify mathematical concepts in the South African context have the same meaning in the Brazilian context. For instance, the phrase 'alternating angles' is understood to have the same meaning around the globe.

Visual mediators

Visual mediators are visible objects that can be utilised to communicate geometric and other mathematical concepts. These visual objects are essential because they can act as classifiers of colloquial and mathematical discourse. In mathematics discourse, visual mediators are only created to mediate a specific discourse that has developed or will develop at a particular time and in a specific space. In addition, visual mediators are signals for visual thinking; many prominent mathematicians have relied to a large degree on their visual thinking for success (Lean & Clements, 1981; Shepard, 1978; Yurmalia & Herman, 2021). These mediators can be used as thinking aids in mathematics discourse and can improve the discourse. The algebraic symbols, notations, and geometric diagrams used in this study are examples of visual mediators. Diagrams play a significant role in geometry; in most cases, they are key to solving geometry problems (Yahya et al., 2022). Sfard (2008) considers mathematical objects as abstract in nature, and the only possible way to reduce the abstractness of these objects is through the use of diagrams or visual data. For example, a point, arc, or line are geometric concepts that can be well conceptualised through representation in visual form. If a geometry Grade 10–11 problem is posed in the English language (i.e. using words), multilingual learners have to engage in various steps, such as comprehending the problem statement first, engaging in individualisation ('thinking as communication'), arriving at a sound understanding of the statement so that they will be able to draw a correct diagram that will assist them in solving the problem. This view is supported by Samkoff et al. (2012),

who argue that 'drawing diagrams is commonly cited as a heuristic for mathematics problem-solving that learners should engage in' (p. 50). This shows that diagrams have the potential to enhance learners' learning of mathematics and geometric understanding, as well as to develop their problem-solving abilities – all of which are fundamental to mathematics education. A visual mediator – such as a diagram – can provide multilingual learners with discursive prompts, enabling them to recall specific knowledge and ways of mathematical problem-solving.

Routines

Routines play an important and special role in mathematical exercises. According to Sfard (2008), routines 'are repetitive patterns characteristic of given discourse' (p. 134). Routines are repetitive and well-expressed discursive patterns that may involve the process of mathematical generation or completion of specific procedures (Berger, 2013). Routines are regulated by mathematical rules, such as what counts as the definition of a cyclic quadrilateral, what constitutes a mathematical proof, or even how to calculate the area of a kite. Once a narrative has been endorsed mathematically, it can be used as a routine or reason to endorse other mathematical narratives. Different patterned ways of communication exist for geometry, and multilingual learners can use any method they choose when solving problems. The routines they choose to use demonstrate their ability to visualise the geometry problem statement, how visuals influence their understanding during problem-solving questions, and, ultimately, the importance of visualising can be assessed by how multilingual learners tackle geometry word questions in terms of the routines used. This can be observed in the way multilingual learners use words and visual mediators to derive new or substantiate existing mathematical narratives. Furthermore, the context, the teacher, the classroom environment, the language, and other factors may influence multilingual learners' discourse about their thinking during geometry problem-solving (Barwell, 2018).

Narratives

In discourse, a narrative is an idea that must be discussed and endorsed mathematically, and once supported, it is regarded as a theory (Sfard, 2008). Sfard (2008) defines a narrative as:

> [*A*]ny sequence of words or utterances framed as a description of objects, or relations between objects, or process with or by objects, that is subject to endorsement or rejection with the help of discourse-specific substantiation procedures. (p. 134)

This means that a narrative is a structured account rather than a random set of statements. For example, 'the sum of angles in a triangle always adds up to 180°'. It is a sequence of words forming a mathematical statement. Furthermore, this means that a narrative must be evaluated within a discourse. The truth of a narrative is endorsed when it is true and rejected when truth cannot be recognised. In geometry, examples of endorsed narratives are a theorem, definition, axiom, postulate, property, or theory because they are derived using mathematical laws. For example, the narrative 'the sum of the interior opposite angles of a cyclic quadrilateral is 180°'. It is endorsed to describe the interior opposite angles of a geometric shape that touches the circumference four times. Different narratives can be used in isolation or in conjunction with others to solve various mathematical problems. Narrative can be produced using words, mediators, and routines. The objective of mathematics is to produce endorsable narratives through the process of alienation in objectification (Sfard, 2008). This implies that the kind of narrative that is produced by a multilingual learner using mathematical rules and methods during geometry problem-solving will demonstrate how visuals influence their understanding.

The connection between commognition and Polya's four steps of problem-solving

A brief explanation of the link between the two theories as perceived in the study suffices. The link between the two theories was explained based on the type of discourse participation provided by multilingual learners during geometry problem-solving questions. The focus was based on how the kind of discourse given by the participants affected the process of problem-solving as provided by Polya (1945). Firstly, the focus was on identifying the relationship between the discourse and the process. The focus was on the success or failure of multilingual learners in solving both word problem statements and problems with diagrams, because this would raise awareness of what influenced the discourse, enabling the participants to solve or fail to solve geometry problems in relation to Polya's model. Based on the theoretical analysis, the failure of most participants to engage fully with Polya's four steps to solve geometry questions was due to their inability to understand the geometry vocabulary, which, in turn, influenced their understanding of visuals. This analysis revealed a significant role these theories can play in improving multilingual learners' comprehension of mathematical language (word use) in geometry, thereby enhancing their thinking and fostering active participation during problem-solving. According to Naidoo and Kampofu (2020), thinking becomes evident when learners can articulate their own understanding during the learning process, thereby promoting active participation. In analysing the two theories, the first step in Polya's model is understanding the problem. In this case of geometry, this means understanding the geometry vocabulary, such as chord, arc, tangent, angles, and diameter. The failure to understand any geometry problem is related to a poor understanding of the above-mentioned words. In commognition, Sfard (2008) asserts that mathematics, on its own, is a language due to its word use. In this study, four elements were utilised: word use, visual mediators, routines, and narratives. Furthermore, this step aligns with two key aspects of Sfard's commognition theory: word use and visual mediators. For word-based questions, word use plays a critical role in learners' comprehension. In contrast, for questions accompanied by diagrams, learners integrate their understanding through visual mediators (such as

diagrams) to develop a more precise conceptual grasp. Bautista et al. (2015) argue that using visual tools (visual mediators) in a structured and complementary manner can significantly enhance learners' understanding of mathematical concepts (word use) and relationships during problem-solving. This relationship was clearly recognised even by participants who successfully solved the geometry problems. As a result, the failure to understand the word use resulted in multilingual learners finding it challenging to follow or construct coherent mathematical narratives due to their poor understanding of geometry vocabulary. The above finding reveals an essential gap in the literature that requires attention, which hinders learners' visualisation skills related to the mathematical vocabulary. Hence, visualisation is regarded as a critical aspect for enhancing learners' understanding (Lim et al., 2020), particularly for learners who may experience challenges in learning mathematics in a language that is not their native language. The second relationship was analysed based on step two of Polya's steps (devising the plan), in relation to how word use and diagrams (visual mediators) influence the planning of the participants in selecting the correct procedure (routine) and another commognition tenet. It was found that participants' devising of a proper plan was based on the influence of visuals and understanding of the problem statement, which relates to word use. This was the significant finding from the analysis of these two theories, which underscores the importance of visuals during problem-solving of geometry questions, which can be applied by teachers in trying to help multilingual learners comprehend geometry questions easily. This finding is also supported by the study by Govender and Amevor (2025), who support the view that learning geometry should incorporate visual representations, which enable learners to think geometrically and communicate geometric reasoning. Furthermore, learners plan by selecting from familiar routines, which depends on their prior exposure to mathematical vocabulary. For example, a multilingual learner who struggled to solve geometry word problems did so because they lacked understanding of word choice; as a result, no proper planning was done to solve the problem. In contrast, those who understood the geometry vocabulary managed to devise an appropriate plan and endorsed the narrative. Devising a plan or routine in geometry (explorative or ritualistic) comprises predicting a reasonable mathematical narrative: how applying specific rules, theorems, and properties of geometric shapes could lead to a solution. As a result, if a multilingual learner cannot formulate a coherent mathematical narrative due to a lack of understanding of vocabulary (word use) and diagrams (visual mediators), the planning is hindered. This finding concurs with Mudaly (2021), who argues that diagrams are visual tools that may help individuals make meaning of mathematical word problems (word use). Thus, it is necessary to find more ways to improve multilingual learners' understanding of both visuals and mathematical vocabulary. In addition, Essien (2010) and Riccomini et al. (2015) found that, in multilingual settings, understanding mathematical word problems is challenging, especially for learners who are still learning the language of instruction while simultaneously mastering mathematical language. Hence, this study focuses on the potential significance of both Sfard's (2008) commognition and Polya's (1945) steps of problem-solving in enhancing multilingual learners' understanding and spatial reasoning during geometry problem-solving. In support of this, Polya (1957) asserts that learners explore and assess their understanding before interacting with mathematical problems. The third relationship was analysed in step three of the Polya steps (carrying out the plan). Here, multilingual learners were expected to execute the selected routine (explorative or ritualistic), such as calculating angles and constructing diagrams. From the findings, it was noted that a break in commognitive routines, such as misapplying geometric properties or constructing wrong diagrams, derails execution. A significant finding was that this demonstrated the importance of commognitive tenets during geometry problem-solving in relation to Polya's third step. These findings concur with Van Hiele's (1999) theory of levels of understanding geometry, which explains that learners who have gone through the successive order can thoroughly compare different axioms and use these definitions and axioms in making deductive reasoning. In the context of this article, the analysis shows that understanding commognitive tenets improves geometric understanding, enabling them to engage in Polya's (1945) four steps of problem-solving. For those participants who understood the geometry concepts well, it was easy for them to follow the plan of choosing the correct algorithm and applying the correct theorems and rules when solving geometry questions. Through the analysis of the two theories, it is easy to track what delays multilingual learners' success and what facilitates their progress from one step to another when solving geometry questions. The last relationship was analysed based on the previous step of Polya's steps (looking back; checking and reflecting). At this point, multilingual learners are expected to review their solutions for consistency within the mathematical narrative, like verifying that the sum of the angles of a triangle adds up to 180° and opposite angles of a cyclic quadrilateral are supplementary. Due to a lack of understanding of geometry vocabulary, multilingual learners are limited in their ability to articulate whether their solutions make sense or not. Precise terminology and clear visuals assist in communicating and evaluating the solution. Multilingual learners struggle to explain or justify their solutions, even if they are correct, due to linguistic or symbolic limitations. Notably, this step integrates the two elements of commognition: narratives and routines. Learners apply geometry theorems and axioms (narratives) to solve problems and develop systematic approaches (routines) during problem-solving. This analysis of both theories underscores the importance of using commognitive tenets, which are regarded as mathematical tools that enhance not only problem-solving but also the active, participatory involvement of learners in the teaching and learning of mathematics. Komatsu and Jones (2020) argued that learners' competence in using mathematical tools not only supported spatial and geometric thinking but also enhanced meaningful learning and participation in the mathematics classroom.

Table 1 represents the summary of the relationship commognition tenets and Polya's four steps of problem-solving discussed above.

Implications for the study

The theoretical analysis of difficulties faced by multilingual learners in the study stems from a disconnect between cognitive elements and problem-solving steps, for instance:

- Limited understanding of word use affects the understanding of the problem.
- Weak visual abilities hinder both understanding and execution of solutions.
- Inadequate routines limit planning and execution.
- Underdeveloped mathematical narratives reduce the ability to reflect on or verify solutions.

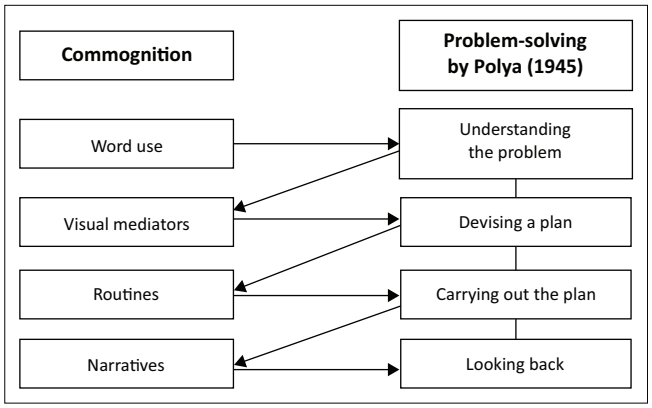
By explicitly aligning instruction to develop these four elements, teachers can better support multilingual learners at each step of the problem-solving process. Figure 2 represents the above relationships based on the theoretical analysis of this literature.

Conclusion

Upon conducting this literature review, this article highlights how multilingual learners encounter challenges in geometry problem-solving due to language-related obstacles, especially with mathematical vocabulary and discourse. A clear correlation emerges by analysing these challenges through Sfard's (2008) commognitive framework and aligning them with Polya's (1945) four steps of problem-solving. The findings show that a multilingual learner's ability to engage fully in Polya's four steps successfully during problem-solving depends on the commognitive tenets, according to Figure 2. Thus, in conclusion, we discuss the key role of the theories of commognition and Polya's problem-solving. We discuss how the findings from this study incorporated commognition and Polya's four steps of problem-solving to generate new knowledge. We use Figure 2 as a guide for the discussion, which enables us to illustrate how multilingual learners' discourse participation can be used to enhance their engagement during problem-solving using Polya's problem-solving steps. The discussion is guided by commognitive tenets and multilingual learners' performance during geometry problem-solving in this study. Thus, we discuss how these tenets influence multilingual learners' movement from the first step to the fourth step. Furthermore, the discussion is guided by the success or failure of multilingual learners in geometry problem-solving, as revealed by the findings.

TABLE 1: A relationship between commognitive tenets and Polya's steps of problem-solving.

Polya's step	Relevant commognition elements
1. Understanding	Word use, visual mediators, narratives
2. Devising a plan	Routines, narratives
3. Carrying out the plan	Routines, visual mediators
4. Looking back	Narratives, word use, visual mediators



Note: Please see the full reference list of the article, Hlongwana, P., & Mudaly, V. (2026). Problem-solving in geometry: A multilingual perspective through commognitive and Polya's steps of problem-solving. *Pythagoras, 47*(1), a866. https://doi.org/10.4102/pythagoras.v47i1.866, for more information.

FIGURE 2: A commognitive analysis of multilingual learners during geometry problem-solving through Polya's four problem-solving steps.

Understanding the problem

The understanding of the geometry problem statement relies on the geometry vocabulary, which is referred to as word use in commognition. Multilingual learners must comprehend both the language of instruction and the geometry vocabulary. This could enable them to understand the entire geometry problem and what is required of them. Stephany (2021) concurs that at the beginning of the problem-solving process, the problem statement must be understood very well. Furthermore, understanding the rules of theorems and the terminology of geometry shapes, such as tangent, chord, diameter, circumference, and supplementary angles, is used in geometry learning and teaching, and in understanding geometry problems. When multilingual learners are only given problem statements, their ability to visualise the problem and construct a diagram depends on their understanding of the words used. According to the literature, a lack of knowledge of geometry vocabulary (word use) hinders multilingual learners' visualisation abilities and hampers their understanding of geometry problems (Essien, 2010; Lariviere et al., 2025). Thus, multilingual learners who cannot understand the vocabulary of geometry fail to understand the geometry problem. In line with this study's findings, Vula and Kurshumlia (2015) confirm that, for most learners, a lack of mathematical vocabulary is a serious challenge in solving mathematical word problems. In conclusion, based on the analysis of both theories, understanding word use is significant for solving geometry problems. Therefore, multilingual learners must be assisted in developing and advancing their mathematical vocabulary and register to master not only the mathematical content but also the mathematical genre.

Devising the plan, carrying out the plan, and looking back

In mathematical problem-solving, routines reflect the learner's procedural knowledge, such as understanding the words used in the problem, applying theorems, and understanding visual mediators. Knowledge of word choice and visuals is critical for devising and executing the plan.

At this stage, in this study, multilingual learners who failed to understand the problem due to a lack of mastery of geometry vocabulary had weak visualisation skills in geometry word problems and in problems with diagrams. The findings revealed that, for multilingual learners to devise and execute the plan, an understanding of geometric vocabulary is required, which enhances their visualisation abilities and enables them to develop the plan (exploratory or ritualistic). Those multilingual learners who demonstrated a good understanding of geometry vocabulary (word use) were able to construct accurate diagrams (visuals) and devise proper plans in both ritualistic and exploratory ways. In contrast, those multilingual learners who failed to understand the geometry problem statements could not formulate and execute adequate planning based on their misconceptions. These findings contribute to the existing literature by bringing new evidence from multilingual contexts and demonstrating the applicability of Sfard's commognitive tenets and Polya's problem-solving steps to geometry understanding. On the other hand, it provides concrete guidelines for navigating learners' misconceptions in geometry learning using the two integrated frameworks. Lastly, narratives tie together logic and reasoning, which are critical in planning and reflecting on solutions. The ability to reflect on the solution requires multilingual learners to understand what informed their routines, enabling them to understand and verify their solutions. By knowing that a shape is a triangle, multilingual learners can verify their angles by proving that they add up to 180°. This stage of problem-solving requires multilingual learners to be clear about the narratives they provide before and after executing any routine so that they can verify their solutions. The narratives given during geometry problem-solving rely on understanding the problem statement and the influence of visuals, which inform the choice of any routine to be executed. Multilingual learners who were not fluent in geometry vocabulary demonstrated many misconceptions, providing incorrect narratives and angles that relied on the appearance of diagrams rather than understanding what was presented. As a result, they were unable to return and verify their solutions. From a cognitive perspective, these findings demonstrate that most multilingual learners rely on procedural memory rather than conceptual understanding. According to Hlongwana et al. (2025), routine-based learning develops short-term accuracy but not long-term conceptual understanding. Similarly, Campbell and Bean (2025)'s findings suggest a continuing decline in students' self-confidence in mathematics, highlighting weak conceptual structures that contributed to uncertainty even in correct responses. These findings suggest that in multilingual contexts, challenges in mathematics learning are both cognitive, arising from fragmented conceptual structures, and pedagogical, stemming from limited understanding of mathematical vocabulary and metacognition reflections. In conclusion, the cognition framework and Polya's four steps of problem-solving offer complementary strengths for supporting multilingual learners in geometry: while the cognition framework foregrounds the relationship between language, prior knowledge, and conceptual understanding, making visible the often-hidden cognitive and linguistic demands of geometric reasoning, Polya's structured problem-solving steps provide a clear, procedural framework that guides learners through mathematical activities. Significantly, the cognition framework affords insights into meaning-making, discourse, and misconceptions that Polya's model does not explicitly address. In contrast, Polya's framework contributes a disciplined, transferable strategy for tackling problems that the cognition perspective alone does not operationalise. When integrated, these frameworks enable a more universal pedagogy that simultaneously supports how multilingual learners think, communicate, and act in geometry problem-solving contexts. In summary, this article confirms that the four elements of commognition by Sfard (2008) and Polya's (1945) four steps of problem-solving complement each other during the geometry problem-solving process due to their strong connection, which serves as a key tool for identifying misconceptions, developing reasoning skills, building conceptual understanding, and designing personalised pedagogical strategies. It offers a more comprehensive approach to understanding learning and helps improve the quality of geometry education in Grades 10–12, especially in multilingual contexts.

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Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

CRediT authorship contribution

Phumlani Hlongwana: Writing – original draft. Vimolan Mudaly: Supervision; Writing – review & editing. Both authors reviewed the article, contributed to the discussion of results, approved the final version for submission and publication, and take responsibility for the integrity of its findings.

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Data availability

The data that support the findings of this study are available on request from the corresponding author, Phumlani Hlongwana.

Disclaimer

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