A law of time dilation proportionality in Keplerian orbits

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We examine the Lorentz factor and the Schwarzschild solution in relation to the estimation and verification of time dilation in particular from Global Positioning System data. As a result, we detect the possible occurrence of a proportionality between the time dilation effects of special and general relativity in free-fall motion in Keplerian orbits. This observation is then proved mathematically. The results show that gravitational time dilation during free fall in Keplerian orbits must be exactly double that caused in special relativity due to linear velocity. We propose that a law be enunciated in view of the proof provided, and of the experimental and technological verification of time dilation effects during the past six to seven decades. The importance of this finding as a universal phenomenon and in the further development of stable clocks and satellite technology is highlighted.

Introduction

In this paper we examine the estimation and experimental verification of time dilation as a result of which we prove that, in Keplerian or free-falling orbital motions resulting from the inverse-square gravitational attraction of a central body, time dilation due to special relativity (SR) must be exactly half that due to the gravitational effect. Because time dilation as a result of relativistic effects has been proved in several experiments and technical applications, the conclusion derived in this paper deserves to be enunciated as a law.

In special relativity theory, time dilation[1,3] is described by the Lorentz transformations. The Schwarzschild solution describes gravitational time dilation in general relativity (GR).[2,3] These two tools have been of vital and fundamental importance in arriving at the law of proportionality of time dilation and we dwell on them in some detail.

Values of time dilation predicted by the Lorentz factor and the Schwarzschild gravitational formula have been confirmed experimentally. Particle accelerators have routinely carried out experimental tests of the time dilation of special relativity since the 1950s. Early SR time dilation tests, which confirm Einstein's prediction, include the measurement[4,5] of the Doppler shift of the radiation emitted from cathode rays, the direct observation of the transverse Doppler shift,[6] and the accurate time dilation recorded in the decay of elementary particles.[7]

Recent experimental verification of SR time dilation includes the significantly improved test performed with laser spectroscopy on fast ions at the heavy-ion storage-ring in Heidelberg at $v = 0.0064\,c$, where c is the speed of light.[8] The result confirms the relativistic Doppler formula. More recently, even more sensitive measurements of time dilation using the Heidelberg storage-ring confirmed time dilation with unprecedented accuracy at $v = 0.0065\,c$.[9]

Gravitational time dilation has been confirmed by the classical example of the measurement of red shifts,[10] with original results within 10% of the predictions of general relativity, which were later[11] improved to 1%. The fact of SR and GR combined time dilation effects was demonstrated using cesium clocks travelling in commercial airliners.[12] These results were recently confirmed[13] in a trial using technologically improved cesium clocks with greater accuracy, to within 4% of the predictions of relativity. Other recent work includes the measurement of the frequency shift of radio photons to and from the Cassini spacecraft as they passed by the Sun, which agreed with the predictions of general relativity.[14]

The Global Positioning System (GPS) and Glonass can be considered as stable clocks in tests involving special and general relativity. It has been emphasized[15] that the GPS provides a fascinating menu of applications of special and general relativity. The use of stable clocks in space navigation technologies depends greatly on the relativity predictions of Einstein and on the use of the Lorentz factor and the Schwarzschild solution.

A recent paper[16] aptly states that relativity has already acquired the status of an applied technology in daily life, a conclusion illustrated by the role of the GPS in the success of the German highway-toll system, which is a business worth several billion euros annually. In addition, international atomic time is defined by comparing and averaging the times provided by more than 200 atomic clocks distributed worldwide. The average value obtained is more accurate than any individual value. Comparisons between individual clocks and one on board a GPS satellite may require uncertainty of less than a few nanoseconds (ns).[16] However, relativistic effects much greater than this have to be corrected. Timing errors of one nanosecond will lead to positioning errors of the order 30 m.[17]

In this work, the data shown below have the corresponding values:

Speed of light $c = 299\,792\,458$ m/s
Gravitational constant $G = 6.6742 \times 10^{11}$ $m^3s^{-2}k^{-1}$
Mass of the Earth $M = 6 \times 10^{24}$ kg
Radius of the Earth $R = 6\,380\,000$ m.

Data for GPS satellites have been given by various authors,[15–19] some of which are the following:

Altitude = 20 200 km
Orbital radius r =26 580 000 m
Velocity v = 3 888 m/s
SR time dilation = +7260 ns/day
GR time dilation = –45 570 ns/day compared with Earth-based stationary clock
Net time dilation = –38 310 ns/day compared with Earth-based clock.

Computing the reported orbital radius against v of the satellite gives a g value of 0.5687. Using the GM/r^2 formula, the value of g is 0.5668. The GPS has a semi-synchronous orbital period of 11 h 58 min, which is half of the Earth's average rotational period of 23 h 56 min. The GPS's successful demonstration of time dilation resulting from various effects of relativity is a clear proof of the validity of Einstein's relativity predictions.

This paper relies heavily on time dilation on account of its experimental verification in both research and practical applications. Although space contraction has not been experimentally

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verified, there is little doubt that it will be shown to be true once relativistic velocities are realized for long distances. This study explores a new perspective of the Lorentz factor for determining time dilation and length contraction. Because the scheme developed (SR scheme 1) works well, we will apply it also to the Schwarzschild solution for gravitational time dilation. We shall then examine related approaches for estimating time dilation.

Estimation of time dilation

The Lorentz factor may be derived from a valid Pythagorean Theorem based on light signals. Here we will adopt the thought experiment based on a moving train passing an observer B standing on an embankment. The train moves with velocity v. A light emitter on the floor of the train sends a light signal straight to a mirror fixed on the train's ceiling exactly opposite the emitter, at a distance L from it. To a train traveller (observer A) the light signal bounces off the mirror and straight back to the floor where the emitter is located. For observer B the signal moves obliquely up towards the mirror and obliquely down to the floor. Since the velocity of light is constant in all reference frames, observer A finds the light signal to take time $t = 2L/c$. For observer B on the embankment, the time taken is different and is given by $t' = 2L'/c$. In the interval, the train has moved a distance $t'v$, v being the train's velocity, and t' the time observer B sees the light signal.

The three distances fit into a right-angled Pythagorean triangle and are given by

$$\frac{t'c}{2}, \frac{t'v}{2}, \frac{tc}{2},$$

which gives:

$$\left(\frac{t'c}{2}\right)^2 = \left(\frac{t'v}{2}\right)^2 + \left(\frac{tv}{2}\right)^2,$$

from which we can derive the multiplier factor:

$$\frac{c}{\sqrt{c^2 - v^2}},$$

which in turn can be translated to the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

For the sake of simplification, we shall first restrict our discussion to the binomial expansion of the Lorentz factor, which is:

$$\frac{v^2}{2c^2}$$

for non-relativistic velocities, which Einstein[1] also used. The Schwarzschild solution, below, is meant to apply to space-time in the vicinity of a non-rotating massive object:

$$t = \frac{t_0}{\sqrt{1 - \frac{2GM}{Rc^2}}},$$

where t is the time interval measured by a clock situated very far away from a central mass. If the clock is at the Earth's surface or not very far from it, we use the gravitational time dilation formula:

$$t = \frac{t_0}{\sqrt{1 - \frac{2gR}{c^2}}}.$$

Employing a binomial expansion of the gravitational time dilation expression, the first approximation to the time expression, ignoring subsequent ones as they are negligible, is:

$$\frac{gr}{c^2}.$$

This formula is a good approximation for estimating time dilation in GRT for r values at the Earth's surface, and applies also to satellites orbiting the Earth.

Based on the Lorentz factor (SR scheme 1), we will initially develop a linear sequence of ideas and calculations, which will lead us eventually to the proportionality of SR and GR time dilation effects. To start with, without making any fundamental change, we propose modifying the formula

$$\frac{v^2}{2c^2} \text{ to } \frac{dv}{2c^2},$$

where d is the distance covered by any moving object with velocity v in time t as seen by an external observer in an inertial frame of reference. We will now examine how we can paint a descriptive and logical sequence of ideas on $dv/2c^2$, that enables us to arrive at time dilation and distance contraction in a moving body. In other words, instead of v^2 and c^2, we shall proceed by adopting a linear sequence of mathematical calculations, bypassing the direct use of squares and square roots (SR scheme 1 below). This approach can be adapted to any kind of motion at uniform velocity such as walking, travelling with clocks to synchronize them, to the motion of an aeroplane, or the linear motion of a Glonass or GPS satellite. Irrespective of the kind of object and its uniform motion, we can proceed as follows.

SR scheme 1 and GR scheme 1

The sequence of ideas and their corresponding sequence of calculations are as follows:

1. First, we obtain the value of d, the distance a massive object (x) moves in time t at uniform velocity v.
2. We divide this value by 2.
3. We then divide $d/2$ by c to give the time that light would theoretically take to cover distance $d/2$.
4. By multiplying the time that light theoretically takes, as obtained in 3) above, with v we obtain the theoretical distance that object x would cover during that time.
5. We now calculate the theoretical time light takes to cover that distance: this actually gives us the amount of time dilation in the inertial reference frame of x.
6. We then calculate the distance x would have theoretically covered in that amount of dilated time to obtain the length contraction in distance.

We shall use this scheme to calculate SR time dilation and distance contraction (SR scheme 1) for a GPS satellite moving at 3888 m/s for 11h 58 min to complete one orbital round. We thereby obtain the following data, bearing in mind that the calculated data will be slightly different from the observed data owing to some fluctuations in altitude, in g value and linear velocity due to various factors including the orbital eccentricity of the satellites.

1. Distance covered in one orbital cycle = 43 080 s × 3888 m = 167 495 040 m.
2. Divide by 2 = 83 747 520 m.
3. Divide by c = 0.279351657 s.
4. Multiply by v = 0.27935 × 3 888 = 1086 m.
5. The time light would take to cover that distance: 1086 divided by c = 0.000003623 s, which gives 3623 ns per cycle, or 7246 ns per day of time dilation due to the linear motion of a GPS satellite.
6. Distance contraction in the satellite frame of reference is therefore 0.000003623 × 3888 = 0.014086 m, or 28.171 mm per day.

The sequence of steps and the corroboration of the data obtained here on time dilation with reported observed time dilation data from atomic clocks in the GPS augur well for the scheme developed above. It also gives weight to Einstein's other prediction of special relativity, that dealing with distance contraction, yet to be measured in a real life situation. In this

connection, the last step in the set of procedures above is very likely also to be correct, especially since up to step 5 everything has been experimentally verified. It is only a matter of time and of technological know-how before it can be experimentally measured.

The arguments proposed in the scheme indicate that physical adjustments in time and space, far from being counter-intuitive, can in fact be intuitive. But the main purpose of the scheme is not to determine time dilation and related relativistic parameters; rather, it is meant to indicate whether analysis of the scheme developed above provides new insights into the coherence of Einstein's relativity theories that would add to a better understanding of motion and of the universe generally.

We shall now proceed to use the above sequence of ideas to see whether each step can be made applicable in GR time dilation (GR scheme 1). To do this, we shall apply it to the gr/c^2 expansion of the Schwarzschild solution, and then illustrate the ideas being argued by applying them to GPS. Before doing so, however, we will provide some explanatory notes regarding g and r in the context of this application. Next, in gr/c^2, we will assume that gr can be equated to the term dv we saw in the modified Lorentz factor mentioned earlier. In the present calculation, we shall consider r to mean the distance d, and g is arbitrarily interpreted as being a uniform 'potential' velocity since an object at rest on the Earth's surface would not per se be accelerating; in other words, there is no real rate of change of velocity, but the object would be sensitive to a gravitational field in proportion to the value of g experienced. This is a mere assumption and it is realized that acceleration may be fundamentally a difficult phenomenon to circumscribe.

One can also argue that a body at rest in a gravitational field, that is, one which does not actually change its r value with respect to the central body exerting the gravitational force, accumulates time dilation values uniformly with time just as a body in uniform motion in SR does. The scheme in GR, given for the sake of comparison, therefore becomes the following for an object at rest at sea level:

1. We assume r to be equivalent to d, that is, equal to 6 380 000 m.
2. Point 2 in the scheme is not applicable in GR.
3. Calculate the time light would theoretically take to cover that distance, which is 0.021281389 s.
4. Calculate how far the object would have theoretically covered in that time, if it were moving at a constant velocity of 9.81 m/s, which is 0.208770428 m.
5. Light would theoretically take 0.6963831918 ns to cover the distance. This is the time dilation value, per second, due to GR for an object at sea level. The total dilation value for a day = 60 000 ns.

Applying the procedure to GPS, we obtain the following for a satellite at a radius of approximately 26 580 000 m completing two orbital cycles per day, each cycle taking 43 080 s. The gravitational attraction at that point is 0.5687 m/s.

1. The value of d is 26 580 000 m.
2. Not applicable in GR.
3. The time that light would theoretically take to cover that distance = 0.088661336 s.
4. The distance that the GPS satellite would have theoretically covered in that time if it were moving at a constant velocity of 0.5687 m/s = 0.0886613 m × 0.5687 m/s = 0.050421702 m.
5. The time that light would take to cover this distance = 0.1681886943 ns/s. Accumulated time dilation for a day = 14 491 ns. The daily difference between a clock at sea level and in the GPS = 60 000 – 14 491 ns = 45 509 ns, which means that a clock in the GPS runs faster than a clock on the Earth's surface by 45 509 ns per day due to the Earth's gravitational field.

The net calculated time dilation therefore is 45 509 – 7246 ns, or 38 263 ns, compared with a stationary clock on Earth. Although apparently an unorthodox approach, the scheme has a fundamental value for it indicates the occurrence of a proportionality. The SR time dilation (SR scheme 1) for a day of 7246 ns, if multiplied by 2, gives 14 492 ns, which is practically identical to the 14 491 ns as obtained for GR time dilation. The above schemes seem to indicate that in terms of actual estimation the most pertinent difference between schemes SR 1 and GR 1 is that the latter produces twice the time dilation value of SR.

GR to SR proportionality

Einstein's theories of special and general relativity incorporate a physical space-time—it certainly makes sense that in SR it is different from that of GR. We have earlier developed a scheme to show that the application of the Lorentz factor and of the Schwarzschild solution to the determination of time dilation appears to have some common features. We shall now proceed further and get close to a proof of a mathematically and therefore structurally proportionate basis for the space-time description in SR and GR.

The obvious differences in the terms, $v^2/2c^2$ for SR and gr/c^2 for GR, appear to fit well with the contextual differences between SR, which is apparently outside any gravitational influence, and GR where gravity has a central role. We shall now see that another system of formulae can be applied to the calculation of time dilation in GR and SR, which will confirm that there must be a proportionality between them. In the case of an object at rest on the Earth's surface, we calculated gravitational time dilation of 0.69638 ns per s or 60 000 ns per day (GR scheme 1) using gr/c^2. For the same time period, a GPS satellite in orbit with r = 26 580 000 m and with g = 0.5687 m s^{-2}, we obtained a time dilation of 0.16818869 ns per s, or 14 491 ns per day, again using gr/c^2. However, for a massive body in Keplerian orbital motion,

$$v^2 = g \times r\,,$$

so that, using GPS satellite data, we get:

$$v = \sqrt{(0.5687 \times 2658000)}\ \text{m/s},$$

$$= 3887.9\ \text{m/s},$$

which is very close to the reported velocity of 3 888 m/s. Calculating GR time dilation, the two formulas shown below will give identical values:

$$\frac{gr}{c^2} = \frac{v^2}{c^2}\,.$$

Since the SR velocity time dilation formula $v^2/2c^2$ gives estimates exactly similar to those of the GR formula v^2/c^2 , the calculated time dilation in GR (GR scheme 2) should be exactly twice that of SR, as shown below for a GPS satellite:

Calculated GPS GR time dilation using v^2/c^2 = 3887.9 m/s × 3887.9 m/s divided by c^2 = 14 490.86 ns per day

Calculated GPS SR time dilation (SR scheme 1) = 7246 ns per day

Reported GPS SR time dilation = 7260 ns.

The calculated gravitational time dilation of an object in circular orbital free fall is therefore exactly twice that of SR time dilation for the same orbital velocity. Consequently, if the observed data are slightly different, they would indicate some slight flaws or inaccuracies in the computation of the data.

We shall again first base our SR time dilation estimation on the use of the binomial expansion of the Lorentz factor, which, as shown earlier, gives us 7246 ns per day for a linear velocity of 3888 m/s for a GPS satellite in orbit. For an object at rest on the Earth's surface, because it has no linear orbital motion, there is

no estimation required. Now let us suppose that an object at rest on the Earth's surface, if the Earth were suddenly to contract in size to that of a tiny black hole, had been in orbital motion due to free fall. We would then get a circular velocity of:

$\sqrt{(9.81 \times 638000} = 7911.345161$ m/s.

Using the formula $v^2/2c^2$, we derive an SR time dilation value of 29 959.69 ns, very nearly half the GR time dilation value of 60 000 ns that we obtained earlier for a stationary clock at sea level.

Conclusion

We have succeeded in demonstrating a proportionality between SR and GR based on both theoretical reasoning and by verification of actual GPS time dilation data. The results produced in this paper warrant a law of proportionality of Einsteinian time dilation effects in Keplerian orbital motion, which states:

> In Keplerian orbital free fall due to the inverse square gravitational attraction of a central body, SR time dilation effects due to linear velocity and GR time dilation have a proportionality of 1:2. This is irrespective of variation in free-fall linear velocity and the value of g due to orbital eccentricity.

Clearly, SR and GR time dilation effects are partially entangled in Keplerian orbits. An interesting observation arising from this proposed law is therefore that all Keplerian orbital inertial frames of reference in the universe would have a common, inherent uniform property in their space-time basis. Irrespective of their relative motions, as long as they are in free-fall motion in Keplerian orbits, they would all have the same proportionality in terms of their SR to GR time dilation. So we can derive an equation with a constant as follows that is applicable only to time dilation in Keplerian orbital free fall:

$$td_{GR} = ktd_{SR}$$

where k is a constant of value 2, td represents the actual time dilation, GR represents general relativity, and SR refers to special relativity.

This relationship foresees also a potential application towards making clocks on board navigational satellites more stable. From the proportionality law, one may envisage automatically and simultaneously monitoring SR and GR time dilation effects with nano-precision, computed from linear velocity or from combined g and r values.

Finally, in retrospect and with an appropriate method, it could have been possible to arrive at the same basic conclusion from an application of the virial theorem, which connects the average kinetic and potential energies of systems in which the potential is a power of the radius.

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