

**TWO-COMMODITY PERISHABLE INVENTORY SYSTEM
WITH BULK DEMAND FOR ONE COMMODITY**

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ABSTRACT

This article considers a two-commodity continuous review inventory system in which the arriving customers belong to any one of three types, such that type 1 customers demand a single item of the first commodity, type 2 customers demand bulk items of the second commodity, and type 3 customers demand one item of the first commodity and bulk items of the second commodity. The arrivals of all three types of customers are assumed to be a Markovian arrival process (MAP). It is also assumed that the number of items demanded for the second commodity is a random variable. The ordering policy is to place orders for both commodities when the inventory levels are below prefixed levels for both commodities. The lead time is assumed to have a phase type distribution, and the demands that occur during stock out period are assumed to be lost. The joint probability distribution for both commodities is obtained in the steady state case. Various measures of system performance and the total expected cost rate in the steady state are derived. The results are illustrated with numerical examples.

OPSOMMING

'n Tweeprodukstelsel se voorraad vir klante word kontinu hersien. Die vraag na die produktipes word gekenmerk deur klantvoorkeure. Die versoektempo van klante vir voorraad word aanvaar as 'n Markovproses. Aannames word gemaak oor vraaghoeveelhede en aanlooptyd. Die resultate van die ondersoek word voorgehou via syfervoorbeelde.

1. INTRODUCTION

One of the factors that contribute to the complexity of present day inventory systems is the multitude of items stocked, necessitating multi-commodity inventory systems. In early dealing with such systems, many models were proposed with independently established re-order points. But in situations where several products compete for limited storage space, or share the same transport facility, or are produced on (procured from) the same equipment (supplier), the above strategy overlooks the potential savings associated with joint ordering, and hence will not be optimal. Thus coordinated (also known as joint) replenishment reduces the ordering and setup costs, and allows the user to take advantage of quantity discounts, if any. Various models and references may be found in Miller [23], Agarwal [5], Silver [31], Thomstone and Silver [36], Kalpakam and Arivarignan[15], and Srinivasan and Ravichandran [35]. and further references that they contain.

In continuous review inventory systems, Balintfy [6] and Silver [31] have considered a coordinated re-ordering policy, represented by the triplet (S, c, s) , where three parameters, S_i, c_i and s_i are specified for each item i with $s_i \leq c_i \leq S_i$, for a unit sized Poisson demand and constant lead time. In this policy, if the level of the i -th commodity at any time is below s_i , an order is placed for $S_i - s_i$ items; and at the same time, if any other item $j(\neq i)$ with available inventory is at or below its can-order level c_j , an order is placed so as to bring its level back to its maximum capacity S_j . Many subsequent articles have appeared with models involving the above policy. A further article of interest is that of Federgruen et al. [9], which deals with the general case of compound Poisson demands and non-zero lead times.

Throughout the years, the work on methods to solve the joint replenishment problem has been extensive. Readers are referred to the publications of Fung and Ma [10], Goyal [11,12,13], Goyal and Satir [14], Kaspi and Rosenblatt [16], Nilsson et al. [27], Nilsson and Silver [28], Olsen [29], Silver [32], Van Eijs [37], Viswanathan [38,39,40], and Wildeman et al. [41] and references that they contain.

Kalpakam and Arivarignan [15] have introduced an (s, S) policy with a single re-order level s defined in terms of the total number of items in the stock. The policy avoids separate ordering of each commodity. Hence a single processing of orders for both commodities has some advantages in a situation where procurement is made from the same supplies, items are produced on the same machine, or items have to be supplied by the same transport facility.

In the case of two-commodity inventory systems, Anbazhagan and Arivarignan [1,2,3,4] have proposed various ordering policies. Yadavalli et al. [42] have analysed a model with a joint ordering policy and variable order quantities. Sivakumar et al. [33] have considered a two-commodity substitutable inventory system in which the items demanded are delivered after a random time. Sivakumar et al. [34] have considered a two-commodity perishable inventory system with a joint ordering policy.

There are some situations in which a single item is demanded for one commodity and multiple items are demanded for another commodity. For instance, a customer may buy a single razor, or a set of blades, or both. Another example is the sale of a DVD writer and a set of DVDs. It may be noted that the seller would be placing a joint order for both commodities, as these are available from the same source. Moreover, a seller may not be willing to place orders frequently, and may prefer to have one order to replenish stock in a given cycle. These situations are modelled in this work by assuming demand processes that require a single item for one commodity, multiple items for the other commodities, or both commodities, and by assuming a joint re-order for both commodities.

This paper is organised as follows. In section 2, the mathematical model and notations

followed in the rest of the paper are described. The steady state solution of the joint probability distribution for both commodities, the phase of the demand process, and the phase of the lead time process are given in section 3. In section 4, various measures are derived of system performance in the steady state; and the total expected cost rate is calculated in section 5. Section 6 presents the cost analysis of the model using numerical examples.

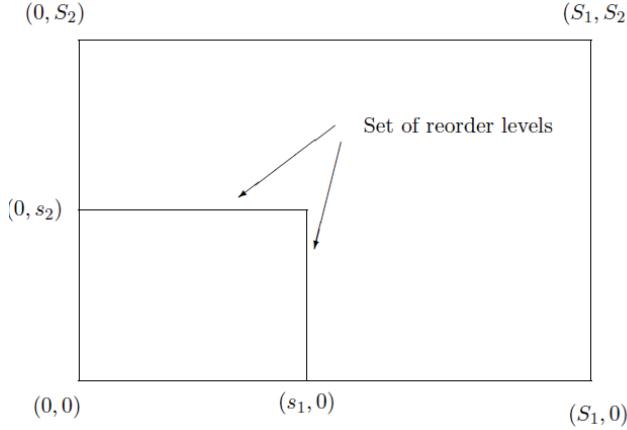


Figure 1: Space of inventory levels

Notations

$\mathbf{0}$: zero matrix

I : an identity matrix

$$H(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$E_i = \{1, 2, \dots, i\}$$

$$E_i^0 = \{0, 1, \dots, i\}$$

e = a column vector of ones

2. MODEL DESCRIPTION

Consider a two-commodity perishable inventory system with the maximum capacity S_i units for the i -th commodity ($i = 1, 2$). Assume that the demand for the first commodity is for a single item and the demand for the second commodity is for bulk items. An arriving customer may demand only the first commodity, or only the second commodity, or both. The number of items demanded for the second commodity at any demand point is a random variable Y with a probability function $p_k = \Pr\{Y = k\}$, $k = 1, 2, 3, \dots$. The three types of demand for these two commodities occur according to a Markovian arrival process (MAP). The life time of each commodity is exponential with parameter γ_i ($i = 1, 2$). The re-order level for the i -th commodity is fixed at s_i ($1 \leq s_i \leq S_i$), and the ordering quantity for the i -th commodity is Q_i ($= S_i - s_i > s_i + 1$) items when both the inventory levels are less than or equal to their respective re-order levels. Assume that demands during stock-out period are lost, as well as unsatisfied demands. The requirement $S_i - s_i > s_i + 1$, ensures that after a replenishment the inventory levels of both commodities will always be above the respective re-order levels. Otherwise it may not be possible to place a re-order (according to the policy), leading to a perpetual shortage. That is, if $L_i(t)$ represents inventory level of i -th commodity at time t , then a re-order is made when $L_1(t) \leq s_1$ and $L_2(t) \leq s_2$ (see Figure 1). The time to deliver

the items is assumed to be of phase type (PH) with representation (α, T) of order m_2 . Note that the phase type distribution is defined as the time until absorption in a finite state irreducible Markov chain with one absorbing state. The mean of the phase type distribution (α, T) is given by $\alpha(-T)^{-1}\mathbf{e}$. Let β denote the reciprocal of this mean. That is, $\beta = [\alpha(-T)^{-1}\mathbf{e}]^{-1}$ gives the rate of replenishment once an order is placed. Let T^0 be such that $T\mathbf{e} + T^0 = \mathbf{0}$.

For the demand process, the description of MAP as given in Lucantoni [20] was used. Consider a continuous-time Markov chain on the state space $1, 2, \dots, m_1$. The demand process is constructively defined as follows. When the chain enters a state i , $1 \leq i \leq m_1$, it remains for an exponential time with parameter θ_i . At the end of the sojourn time in state i , there are four possible transitions: with probabilities a_{ij} , $1 \leq j \leq m_1$, the chain enters the state j when a demand for the first commodity occurs; with probabilities b_{ij} , $1 \leq j \leq m_1$, the chain enters the state j when a demand for the second commodity occurs; with probabilities c_{ij} , $1 \leq j \leq m_1$, the chain enters the state j when a demand for both commodities occurs; with probabilities d_{ij} , $1 \leq j \leq m_1, i \neq j$, the transitions correspond to no demand and the state of the chain is j . The Markov chain can go from state i to state i only through a demand. Defining the square matrices D_k , $k = 0, 1, 2, 12$, of size $m_1 \times m_1$ by $[D_0]_{ii} = -\theta_i$ and $[D_0]_{ij} = \theta_i d_{ij}$, $i \neq j$, $[D_1]_{ij} = \theta_i a_{ij}$, $[D_2]_{ij} = \theta_i b_{ij}$ and $[D_{12}]_{ij} = \theta_i c_{ij}$, $1 \leq i, j \leq m_1$. It is seen that $D = D_0 + D_1 + D_2 + D_{12}$ is an infinitesimal generator of a continuous-time Markov chain. Assume that D is irreducible and $D_0\mathbf{e} \neq 0$.

Let ζ be the stationary probability vector of the continuous-time Markov chain with generator D . That is, ζ is the unique probability vector satisfying $\zeta D = 0$, $\zeta\mathbf{e} = 1$.

Let η be the initial probability vector of the underlying Markov chain governing the MAP. By choosing η appropriately, the time origin can be modelled to be:

1. an arbitrary arrival point,
2. the end of an interval during which there are at least k arrivals; or
3. the point at which the system is in a specific state, such as when the busy period ends or begins.

The important case is that of the stationary version of the MAP for $\eta = \zeta$. The constant $\lambda = \zeta(D_1 + D_2 + D_{12})\mathbf{e}$, referred to as the **fundamental rate**, gives the expected number of demands per unit time in the stationary version of the MAP. The quantities $\lambda_1 = \zeta D_1\mathbf{e}$, $\lambda_2 = \zeta D_2\mathbf{e}$ and $\lambda_{12} = \zeta D_{12}\mathbf{e}$, give the arrival rate of demand for first commodity, for second commodity, and for both respectively. Note that $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

For further details on MAP and phase-type distributions and their usefulness in stochastic modelling, the reader may refer to Chapter 2 in Neuts [24], Chapter 5 in Neuts [25], Ramaswami [30], Lucantoni [20, 21], Lucantoni et al. [22], Latouche and Ramaswami [17], Li and Li [19], Lee and Jeon [18], Chakravarthy and Dudin [8], and references therein for a detailed introduction of the MAP and phase-type distribution. Some recent reviews can be found in Neuts [26] and Chakravarthy [7].

Let $J_1(t)$ and $J_2(t)$ respectively denote the phase of the demand process and the phase of

the lead time process. Then the stochastic process $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ has the state space,

$$\begin{aligned}\Omega = & \left\{ (i_1, i_2, i_3, 0), i_1 \in E_{S_1} \setminus E_{S_1^0}, i_2 \in E_{S_2} \setminus E_{S_2^0}, i_3 \in E_{m_1} \right\} \\ & \cup \left\{ (i_1, i_2, i_3, 0), i_1 \in E_{S_1} \setminus E_{S_1^0}, i_2 \in E_{S_2^0}, i_3 \in E_{m_1} \right\} \\ & \cup \left\{ (i_1, i_2, i_3, 0), i_1 \in E_{S_1^0}, i_2 \in E_{S_2} \setminus E_{S_2^0}, i_3 \in E_{m_1} \right\} \\ & \cup \left\{ (i_1, i_2, i_3, i_4), i_1 \in E_{S_1^0}, i_2 \in E_{S_2^0}, i_3 \in E_{m_1}, i_4 \in E_{m_2} \right\}\end{aligned}$$

From the assumptions made on the demand and replenishment processes, it can be shown that $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ is a Markov process on the state space Ω . By placing the sets of state space in lexicographic order, the infinitesimal generator of the Markov chain governing the system, in block partitioned form, is given by

$$[P]_{ij} = \begin{cases} A_i, & j = i, \quad i = 0, 1, \dots, S_1, \\ B_i, & j = i - 1, \quad i = 1, 2, \dots, S_1, \\ C, & j = i + Q_1, \quad i = 0, 1, \dots, S_1, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

where

$$[C]_{ij} = \begin{cases} I_{m_1} \otimes T^0, & j = i + Q_2, \quad i = 0, 1, \dots, S_2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $k = S_1 + 2, S_1 + 3, \dots, S_1$,

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, \quad i = 1, 2, \dots, S_2, \\ D_1 + D_{12} + k\gamma_1 I_{m_1}, & j = i, \quad i = 0, \\ p_{i-j} D_{12}, & j = 1, 2, \dots, i-1, \quad i = 2, 3, \dots, S_2, \\ p_i D_{12}, & j = 0, \quad i = 1, 2, \dots, S_2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$\vec{p}_n = \sum_{i=n}^{\infty} p_i$$

For $k = S_1 + 1$,

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, \quad i = S_2 + 1, S_2 + 2, \dots, S_2, \\ (D_1 + k\gamma_1 I_{m_1}) \otimes \alpha, & j = i, \quad i = 1, 2, \dots, S_2, \\ (D_1 + D_{12} + k\gamma_1 I_{m_1}) \otimes \alpha, & j = i, \quad i = 0, \\ p_{i-j} D_{12}, & j = S_2 + 1, S_2 + 2, \dots, i-1, \quad i = S_2 + 2, S_2 + 3, \dots, S_2, \\ p_{i-j} D_{12} \otimes \alpha, & j = 1, 2, \dots, S_2, \quad \text{or} \\ & j = 1, 2, \dots, i-1, \quad i = 2, 3, \dots, S_2 + 1, \\ p_i D_{12} \otimes \alpha, & j = 0, \quad i = 1, 2, \dots, S_2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $k = 1, 2, \dots, s_1$

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, \quad i = s_2 + 1, s_2 + 2, \dots, S_2, \\ (D_1 + k\gamma_1 I_{m_1}) \otimes I_{m_2}, & j = i, \quad i = 1, 2, \dots, s_2, \\ (D_1 + D_{12} + k\gamma_1 I_{m_1}) \otimes I_{m_2}, & j = i, \quad i = 0, \\ p_{i-j} D_{12}, & j = s_2 + 1, s_2 + 2, \dots, i-1, \quad i = s_2 + 2, s_2 + 3, \dots, S_2, \\ p_{i-j} D_{12} \otimes \alpha, & j = 1, 2, \dots, s_2, \quad i = s_2 + 1, s_2 + 2, \dots, S_2, \\ p_i D_{12}, & j = 0, \quad i = s_2 + 1, s_2 + 2, \dots, S_2, \\ p_{i-j} D_{12} \otimes I_{m_2}, & j = 1, 2, \dots, i-1, \quad i = 2, 3, \dots, s_2, \\ p_i D_{12} \otimes I_{m_2}, & j = 0, \quad i = 1, 2, \dots, s_2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $k = s_1 + 1, s_1 + 2, \dots, S_1$

$$[A_k]_{ij} = \begin{cases} p_i D_2 + k\gamma_2 I_{m_1}, & j = i-1, \quad i = 2, 3, \dots, S_2, \\ p_{i-j} D_2, & j = 1, 2, \dots, i-2, \quad i = 3, 4, \dots, S_2, \\ p_i D_2, & j = 0, \quad i = 1, 2, \dots, S_2, \\ D_0 - (k\gamma_1 + i\gamma_1) I_{m_1}, & j = i, \quad i = 1, 2, \dots, S_2, \\ D_0 + D_2 - k\gamma_1 I_{m_1}, & j = i, \quad i = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $k = 1, 2, \dots, s_1$

$$[A_k]_{ij} = \begin{cases} p_i D_2 + i\gamma_2 I_{m_1}, & j = i-1, \quad i = s_2 + 2, s_2 + 3, \dots, S_2, \\ (p_i D_2 + i\gamma_2 I_{m_1}) \otimes \alpha, & j = i-1, \quad i = s_2 + 1, \\ (p_i D_2 + i\gamma_2 I_{m_1}) \otimes I_{m_2}, & j = i-1, \quad i = 2, 3, \dots, s_2, \\ p_{i-j} D_2, & j = s_2 + 1, s_2 + 2, \dots, i-2, \quad i = s_2 + 3, s_2 + 4, \dots, S_2, \\ p_{i-j} D_2 \otimes \alpha, & j = 1, 2, \dots, i-2, \quad i = s_2 + 1, s_2 + 2, \\ & \quad \text{or} \\ p_{i-j} D_2 \otimes I_{m_2}, & j = 1, 2, \dots, s_2, \quad i = s_2 + 3, s_2 + 4, \dots, S_2, \\ p_i D_2 \otimes \alpha, & j = 0, \quad i = s_2 + 1, s_2 + 2, \dots, S_2, \\ p_i D_2 \otimes I_{m_2}, & j = 0, \quad i = 1, 2, \dots, s_2, \\ D_0 - (k\gamma_1 + i\gamma_2) I_{m_1}, & j = i, \quad i = s_2 + 1, s_2 + 2, \dots, S_2, \\ D_0 \oplus T - (k\gamma_1 + i\gamma_2) I_{m_1} \otimes I_{m_2}, & j = i, \quad i = 1, 2, \dots, s_2, \\ (D_0 + D_2) \oplus T - k\gamma_1 I_{m_1} \otimes I_{m_2}, & j = i, \quad i = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $k = 0$,

$$[A_k]_{ij} = \begin{cases} p_1(D_2 + D_{12}) + i\gamma_2 I_{m_1}, & j = i-1, \quad i = s_2 + 2, s_2 + 3, \dots, S_2, \\ (p_1(D_2 + D_{12}) + i\gamma_2 I_{m_1}) \otimes \alpha, & j = i-1, \quad i = s_2 + 1, \\ (p_1(D_2 + D_{12}) + i\gamma_2 I_{m_1}) \otimes I_{m_2}, & j = i-1, \quad i = 2, 3, \dots, s_2, \\ p_{i-j}(D_2 + D_{12}), & j = s_2 + 1, s_2 + 2, \dots, i-2, \quad i = s_2 + 3, s_2 + 4, \dots, S_2, \\ p_{i-j}(D_2 + D_{12}) \otimes \alpha, & j = 1, 2, \dots, i-2, \quad i = s_2 + 1, s_2 + 2, \\ & \quad \text{or} \\ & j = 1, 2, \dots, s_2, \quad i = s_2 + 3, s_2 + 4, \dots, S_2, \\ p_{i-j}(D_2 + D_{12}) \otimes I_{m_2}, & j = 1, 2, \dots, i-2, \quad i = 3, 4, \dots, s_2, \\ p_i(D_2 + D_{12}) \otimes \alpha, & j = 0, \quad i = s_2 + 1, s_2 + 2, \dots, S_2, \\ p_i(D_2 + D_{12}) \otimes I_{m_2}, & j = 0, \quad i = 1, 2, \dots, s_2, \\ D_0 + D_1 - (i\gamma_2)I_{m_1}, & j = i, \quad i = s_2 + 1, s_2 + 2, \dots, S_2, \\ (D_0 + D_1) \oplus T - (i\gamma_2)I_{m_1} \otimes I_{m_2}, & j = i, \quad i = 1, 2, \dots, s_2, \\ D \oplus T, & j = i, \quad i = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

It may be noted that the matrix C is of order $(Q_1 m_1 + (s_1 + 1)m_1 m_2) \times (S_2 + 1)m_1$, the matrices $B_i, i = s_1 + 2, s_1 + 3, \dots, S_1$, are of order $(S_2 + 1)m_1 \times (S_2 + 1)m_1$, the matrix B_{s_1+1} is of order $(S_2 + 1)m_1 \times (Q_1 m_1 + (s_1 + 1)m_1 m_2)$, the matrices $B_i, i = 1, 2, \dots, s_1$, are of order $(Q_1 m_1 + (s_1 + 1)m_1 m_2) \times (Q_1 m_1 + (s_1 + 1)m_1 m_2)$, the matrices $A_i, i = 0, 1, \dots, s_1$ are of order $(Q_1 m_1 + (s_1 + 1)m_1 m_2) \times (Q_1 m_1 + (s_1 + 1)m_1 m_2)$, and the matrices $A_i, i = s_1 + 1, s_1 + 2, \dots, S_1$ are of order $(S_2 + 1)m_1 \times (S_2 + 1)m_1$.

3. STEADY STATE ANALYSIS

It can be seen from the structure of P that the homogeneous Markov process $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ on the finite state space Ω is irreducible.

Hence, the limiting distribution $\phi_{(i,k,j_1,j_2)} = \lim_{t \rightarrow \infty} \Pr[L_1(t) = i, L_2(t) = k, J_1(t) = j_1, J_2(t) = j_2 | L_1(0), L_2(0), J_1(0), J_2(0)]$

exists. Let

$$\phi_{(i,k,j_1)} = \begin{cases} (\phi_{(i,k,j_1,1)}, \phi_{(i,k,j_1,2)}, \dots, \phi_{(i,k,j_1,m_2)}), & (i, k, j_1) \in F_1, \\ (\phi_{(i,k,j_1,0)}), & (i, k, j_1) \in F_2, \end{cases}$$

where

$$\begin{aligned} F_1 &= \{(i_1, i_2, i_3), i_1 \in E_{s_1}^0, i_2 \in E_{s_2}^0, i_3 \in E_{m_1}\} \\ F_2 &= \{(i_1, i_2, i_3), i_1 \in E_{s_1} \setminus E_{s_1}^0, i_2 \in E_{s_2} \setminus E_{s_2}^0, i_3 \in E_{m_1}\} \\ &\cup \{(i_1, i_2, i_3), i_1 \in E_{s_1} \setminus E_{s_1}^0, i_2 \in E_{s_2}^0, i_3 \in E_{m_1}\} \\ &\cup \{(i_1, i_2, i_3), i_1 \in E_{s_1}^0, i_2 \in E_{s_2} \setminus E_{s_2}^0, i_3 \in E_{m_1}\} \end{aligned}$$

$$\phi_{(i,k)} = (\phi_{(i,k,1)}, \phi_{(i,k,2)}, \dots, \phi_{(i,k,m_1)}), k \in E_2, i \in E_1,$$

$$\phi^{(i)} = (\phi_{(i,0)}, \phi_{(i,1)}, \dots, \phi_{(i,S_2)}), i \in E_1$$

and

$$\Phi = \left(\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(S_1)} \right).$$

Then the vector of limiting probabilities Φ satisfies

$$\Phi P = \mathbf{0} \quad \text{and} \quad \Phi \mathbf{e} = 1.$$

The first equation of the above yields the following set of equations:

$$\Phi^{(i+1)} B_{i+1} + \Phi^{(i)} A_i = 0, i = 0, 1, \dots, Q_1 - 1, \quad (1)$$

$$\Phi^{(i+1)} B_{i+1} + \Phi^{(i)} A_i + \Phi^{(i-Q_1)} C = 0, i = Q_1, \quad (2)$$

$$\Phi^{(i+1)} B_{i+1} + \Phi^{(i)} A_i + \Phi^{(i-Q_1)} C = 0, i = Q_1 + 1, Q_1 + 2, \dots, S_1 - 1, \quad (3)$$

$$\Phi^{(i)} A_i + \Phi^{(i-Q_1)} C = 0, i = S_1. \quad (4)$$

The equations (except (2)) can be recursively solved to obtain

$$\Phi^{(i)} = \Phi^{(Q_1)} \theta_i, \quad i = 0, 1, \dots, S_1, \quad (5)$$

where

$$\theta_i = \begin{cases} (-1)^{Q_1-i} B_{Q_1} A_{Q_1-1}^{-1} B_{Q_1-1} \cdots B_{i+1} A_i^{-1}, & i = 0, 1, \dots, Q_1 - 1, \\ I, & i = Q_1, \\ (-1)^{2Q_1-i+1} \sum_{j=0}^{S_1-i} \left[\left(B_{Q_1} A_{Q_1-1}^{-1} B_{Q_1-1} \cdots B_{s_1+1-j} A_{s_1-j}^{-1} \right) C A_{s_1-j}^{-1} \right. \\ \left. \left(B_{s_1-j} A_{s_1-j-1}^{-1} B_{s_1-j-1} \cdots B_{i+1} A_i^{-1} \right) \right], & i = Q_1 + 1, \dots, S_1. \end{cases}$$

Substituting the values of θ_i in equation (2) and in the normalising condition, it follows that

$$\begin{aligned} \Phi^{(Q_1)} \left[(-1)^{Q_1} \sum_{j=0}^{S_1-1} \left[\left(B_{Q_1} A_{Q_1-1}^{-1} B_{Q_1-1} \cdots B_{s_1+1-j} A_{s_1-j}^{-1} \right) C A_{s_1-j}^{-1} \right. \right. \\ \left. \left. \left(B_{s_1-j} A_{s_1-j-1}^{-1} B_{s_1-j-1} \cdots B_{Q_1+2} A_{Q_1+1}^{-1} \right) \right] B_{Q_1+1} + A_{Q_1} \right. \\ \left. + (-1)^{Q_1} B_{Q_1} A_{Q_1-1}^{-1} B_{Q_1-1} \cdots B_1 A_0^{-1} C \right] = 0, \end{aligned} \quad (6)$$

and

$$\begin{aligned} \Phi^{(Q_1)} \left[\sum_{i=0}^{Q_1-1} \left((-1)^{Q_1-i} B_{Q_1} A_{Q_1-1}^{-1} B_{Q_1-1} \cdots B_{i+1} A_i^{-1} \right) + I \right. \\ \left. + \sum_{i=Q_1+1}^{S_1} \left((-1)^{2Q_1-i+1} \sum_{j=0}^{S_1-i} \left[\left(B_{Q_1} A_{Q_1-1}^{-1} B_{Q_1-1} \cdots B_{s_1+1-j} A_{s_1-j}^{-1} \right) C A_{s_1-j}^{-1} \right. \right. \right. \\ \left. \left. \left. \left(B_{s_1-j} A_{s_1-j-1}^{-1} B_{s_1-j-1} \cdots B_{i+1} A_i^{-1} \right) \right] \right) \right] \mathbf{e} = 1. \end{aligned} \quad (7)$$

From the equation (6), the value of $\Phi^{(Q)}$ may be obtained up to a constant multiplication. This constant can be determined by substituting the value of $\Phi^{(Q)}$ in the equation (7). Substituting the value of $\Phi^{(Q)}$ in the equation (5) we get the values of $\Phi^{(i)}, i = 0, 1, \dots, S$.

4. SYSTEM PERFORMANCE MEASURES

In this section, certain stationary performance measures of the system are derived. Using these measures, the total expected cost per unit time may be determined.

4.1 Mean inventory level

Let η_{I_k} denote the mean inventory level of k -th commodity in the steady state ($k = 1, 2$)

Since $\phi_{(i,j)}$ is the steady state probability vector for the inventory level of first commodity i and the second commodity j , it follows that

$$\eta_{I_1} = \sum_{i=1}^{s_1} \sum_{j=0}^{s_2} j \phi_{(i,j)} \mathbf{e}.$$

and

$$\eta_{I_2} = \sum_{i=0}^{s_1} \sum_{j=1}^{s_2} j \phi_{(i,j)} \mathbf{e}.$$

4.2 Mean re-order rate

A re-order for both commodities is made when the joint inventory level drops to (s_1, s_2) or (s_1, j) , $j < s_2$ or (i, s_2) , $i < s_1$. Let η_R denote the mean re-order rate for both commodities in the steady state, given by

$$\begin{aligned} \eta_R = & \sum_{k=0}^{s_1} \sum_{j=1}^{s_2} \phi_{(k,s_2+j)} \sum_{u=j}^{\infty} p_u (D_2 \otimes \alpha) \mathbf{e} + \sum_{k=0}^{s_2} \phi_{(s_1+1,k)} (D_1 \otimes \alpha) \mathbf{e} \\ & + \sum_{k=1}^{s_1+1} \sum_{j=1}^{s_2} \phi_{(k,s_2+j)} \sum_{u=j}^{\infty} p_u (D_{12} \otimes \alpha) \mathbf{e} + \sum_{k=0}^{s_1} (s_2 + 1) \gamma_2 \phi_{(k,s_2+1)} \mathbf{e} \\ & + \sum_{k=0}^{s_2} (s_1 + 1) \gamma_1 \phi_{(s_1+1,k)} \mathbf{e}. \end{aligned}$$

4.3 Mean shortage rate

Let η_{Sh_i} denote the mean shortage rate of i -th type demand in the steady state ($i = 1, 2, 12$). Consequently

$$\eta_{Sh_1} = \sum_{k=0}^{s_2} \phi_{(0,k)} D_1 \mathbf{e}.$$

$$\eta_{Sh_2} = \sum_{i=0}^{s_1} \sum_{j=0}^{s_2} \phi_{(i,j)} \sum_{k=j+1}^{\infty} p_k D_2 \mathbf{e}.$$

and

$$\eta_{Sh_{12}} = \left(\sum_{k=0}^{s_2} \phi_{(0,k)} D_{12} \mathbf{e} + \sum_{i=0}^{s_1} \sum_{j=0}^{s_2} \phi_{(i,j)} \sum_{k=j+1}^{\infty} p_k D_{12} \mathbf{e} \right).$$

4.4 Mean failure rate

Let the mean failure rate of commodity i in the steady state be denoted by η_{F_i} , ($i = 1, 2$). A failure occurs when any one of the stocked items ceases to function or perishes. Since the rate of failure of a single item is γ_j for commodity j , the rate at which any one of i items for the j -th commodity fail is given by $i\gamma_j$, ($j = 1, 2$). When the process is in state (i, k, j_1, j_2) ,

the rate of failure of any one item of the first commodity is given by $i\gamma_1$ (provided $i > 0$), and the failure rate of any one item of the second commodity is $k\gamma_2$ (provided $k > 0$).

Then we have

$$\eta_{F_1} = \sum_{i=1}^{S_1} \sum_{k=0}^{S_2} i \gamma_1 \phi_{(i,k)} \mathbf{e}.$$

and

$$\eta_{F_2} = \sum_{i=0}^{S_1} \sum_{k=1}^{S_2} k \gamma_2 \phi_{(i,k)} \mathbf{e}.$$

5. COST ANALYSIS

The total expected cost per unit time (total expected cost rate) in the steady-state for this model is defined to be

$$TC(S_1, s_1, S_2, s_2) = c_{h_1} \eta_{I_1} + c_{h_2} \eta_{I_2} + c_s \eta_R + c_{sh_1} \eta_{Sh_1} + c_{sh_2} \eta_{Sh_2} + c_{sh_{12}} \eta_{Sh_{12}} + c_{f_1} \eta_{F_1} + c_{f_2} \zeta_{F_2}$$

where

c_{h_i} : The inventory carrying cost of i -th commodity per unit item per unit time ($i = 1, 2$) .

c_s : Joint ordering cost per order.

c_{f_i} : The failure cost of i -th commodity per unit item per unit time ($i = 1, 2$) .

c_{sh_i} : Shortage cost due to type i demand per unit time ($i = 1, 2, 12$) .

Since the total expected cost rate is only implicitly known, the analytical properties - such as convexity of the total expected cost rate - cannot be carried out in the present form. However, the following numerical examples to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables, are presented.

6. ILLUSTRATIVE NUMERICAL EXAMPLES

Since the total expected cost rate is obtained in a complex form, the convexity of the total expected cost rate cannot be studied by analytical methods. Hence, 'simple' numerical search procedures are used to find the 'local' optimal values for any two of the decision variables $\{S_1, s_1, S_2, s_2\}$ by considering a small set of integer values for these variables. With a large number of numerical examples, it has been found that the total cost rate per unit time in the long run is either a convex function of both variables or an increasing function of any one variable.

The following five *MAPs* for arrival of demands are considered, and it may be noted that these processes can be normalized to have a specific (given) demand rate λ when considered for arrival of demands.

1. Exponential (Exp)

$$H_0 = (-1)H_1 = (1)$$

2. Erlang (Erl)

$$H_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. **Hyper-exponential (HExp)**

$$H_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \quad H_1 = \begin{pmatrix} 9 & 1 \\ 0.9 & 0.1 \end{pmatrix}$$

4. **MAP with negative correlation (MNC)**

$$H_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 25.25 & 0 & 55.75 \\ 55.75 & 0 & 25.25 \end{pmatrix}$$

5. **MAP with positive correlation (MPC)**

$$H_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 55.25 & 0 & 25.75 \\ 25.75 & 0 & 55.25 \end{pmatrix}$$

All the above *MAPs* are qualitatively different in that they have different variance and correlation structures. The first three processes are special cases of renewal processes, and the correlation between arrival times is 0. The demand process labelled as **MNC** has correlated arrivals with a correlation coefficient of -0.1254, and the demands corresponding to the process labelled **MPC** have a positive correlation coefficient of 0.1213. Since Erlang has the least variance among the five arrival processes considered here, the ratios of the variances of the other four arrival processes, labelled **Exp**, **HExp**, **MNC** and **MPC** above, with respect to the Erlang process, are 3.0, 15.1163, 8.1795, 8.1795 respectively. The ratios are given rather than the actual values, since the variance depends on the arrival rate, which is varied in the discussion.

For the lead time distribution, the following three *PH* – distributions are considered. Again these processes can be normalised to have a specific (given) rate β when considered for replenishment.

1. **Exponential (Exp)**

$$\alpha = (1)T = (-1)$$

2. **Erlang (Erl)**

$$\alpha = (1, 0, 0, 0)T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3. **Hyper-exponential (HExp)**

$$\alpha = (0.9, 0.1)T = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 1: In this example, the effect of the demand rate λ , the lead time rate β , the five types of demand processes, and the three types of lead time processes on the optimal values (S_1^*, S_2^*) and the optimal cost rate $TC(S_1^*, 2, S_2^*, 4)$ is illustrated. The following fixed values for the parameters and costs are assumed:

$$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \gamma_1 = 0.8, \gamma_2 = 0.6, p_i = 0.6 * 0.4^{i-1}, i = 1, 2, \dots, \\ c_{h_1} = 0.05, c_{h_2} = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$$

Table 1 gives the optimum values, S_1^* and S_2^* , that minimise the total expected cost rate for each of the five *MAPs* for arrivals of demands considered against each of the three *PHs* for lead times. The associated total expected cost rate values are also given in the table. The lower entry in each cell gives the optimal expected cost rate, and the upper entries correspond to S_1^* and S_2^* . The following observations from Table 1 may be noticed:

1. As λ increases, the optimal total cost rate decreases for all five demand processes and for all three lead time processes. Similarly, as β increases the optimal total cost rate decreases.
2. The optimal total expected cost rate has a higher value for a demand process having an hyper-exponential distribution, and a lower value for the Erlang demand process.
3. The Erlang distributed lead time has a low optimal total cost rate, except for an **HExp** distributed demand process; and **HExp** distributed lead time has high optimal total cost rate, except for **HExp** distributed demand process. For **HExp** distributed demand process this observation reverses - i.e. **HExp** distributed lead time has a low optimal total cost rate, and **Erl** distributed lead time has a high optimal total cost rate.

Example 2: In this example, the effect of the arrival rate λ , the lead time rate β , and the type of arrival and lead time processes on the optimal values (S_1^*, S_2^*) and optimal cost rate $TC(15, S_1^*, 30, S_2^*)$ is illustrated. Assume the following fixed values for the parameters and cost:

$$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \gamma_1 = 0.6, \gamma_2 = 0.5, p_i = 0.55 * 0.45^{i-1}, i = 1, 2, \dots, \\ ch_1 = 0.01, ch_2 = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$$

The optimum values S_1^* and S_2^* that minimise the expected total cost for each of the five *MAPs* for arrivals of demands considered against each of the three *PHs* for lead times is given in Table 2. The associated total expected cost rate values are also given. The lower entry in each cell gives the optimal expected cost rate, and the upper entries correspond to S_1^* and S_2^* . The key observations are summarised below.

		Lead time distribution					
		10			15		
MAP demands distributions	β	Exp	Erl	HExp	Exp	Erl	HExp
	6	Exp	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)
		8.0236	8.0177	8.0277	8.2027	8.1999	8.2047
		Erl	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)
		7.9967	7.9905	8.0009	8.1838	8.1809	8.1858
		HExp	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)
		8.1568	8.1570	8.1567	8.2977	8.2978	8.2976
		MNC	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)
	8	8.0736	8.0680	8.0774	8.2379	8.2352	8.2397
		MPC	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)
		8.1267	8.1214	8.1303	8.2753	8.2727	8.2770
		Exp	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)
		10.5224	10.5175	10.5258	10.8125	10.8101	10.8141
		Erl	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)
		10.4959	10.4908	10.4994	10.7939	10.7915	10.7956
	12	HExp	(17,58)	(17,58)	(17,58)	(17,59)	(17,59)
		10.6604	10.6608	10.6601	10.9104	10.9106	10.9103
		MNC	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)
		10.5717	10.5670	10.5749	10.8470	10.8448	10.8486
		MPC	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)
		10.6260	10.6215	10.6291	10.8852	10.8830	10.8867

Table 1: Total expected cost rate as a function of (S_1, S_2)

1. As λ increases, the optimal total cost rate increases, except for an **HExp** distributed demand process. For an **HExp** distributed demand process, the optimal total cost rate decreases as the demand rate λ increases.
2. When β increases, the optimal total cost rate increases for all combinations of five arrival processes and three demand processes.
3. The optimal cost rate is high in cases where the demand process is an **HExp**, and it is low when the demand process is Erlang.
4. The optimal total cost rate is low when the lead time is **Erl**, except for the **HExp** distributed demand process. For **HExp** distributed lead time, the optimal total cost rate is high, except for **HExp** distributed demand process. For **HExp** distributed demand process this observation reverses - i.e. the **HExp** distributed lead time is associated with a low optimal total cost rate, and **Erl** is associated with a high optimal total cost rate.

Lead time distribution								
MAP Demands Distributions	β	λ	10			15		
			Exp	Erl	HExp	Exp	Erl	HExp
	6	Exp	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)
			7.2328	7.2286	7.2356	7.3624	7.3604	7.3638
	6	Erl	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)
			7.2080	7.2037	7.2111	7.3450	7.3429	7.3464
	8	HExp	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)
			7.3579	7.3598	7.3567	7.4518	7.4527	7.4512
	8	MNC	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)
			7.2787	7.2748	7.2814	7.3947	7.3929	7.3960
	8	MPC	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)	(4,4)
			7.3282	7.3245	7.3307	7.4296	7.4279	7.4309
	8	Exp	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)
			9.4935	9.4902	9.4957	9.7144	9.7129	9.7155
	8	Erl	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)
			9.4694	9.4660	9.4717	9.6977	9.6961	9.6988
	8	HExp	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)
			9.6194	9.6211	9.6183	9.8030	9.8038	9.8025
	8	MNC	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)
			9.5381	9.5351	9.5402	9.7455	9.7441	9.7465
	8	MPC	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)	(3,4)
			9.5876	9.5847	9.5895	9.7799	9.7786	9.7809

Table 2: Total expected cost rate as a function of (s_1, s_2)

Example 3: Next, consider the impact of c_{f_1} and c_{f_2} on the total expected cost rate. For this, consider the following values for the parameters and costs:

$$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \lambda = 8, \beta = 0.5, \gamma_1 = 0.6, \gamma_2 = 0.5, p_i = 0.55 * 0.45^{i-1}, i = 1, 2, \dots, c_{h_1} = 0.01, c_{h_2} = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1.$$

The graphs of the total expected cost rate as a function of c_{f_1} and c_{f_2} for the three lead time processes and the five demand processes are shown in Figures 2 to 6. In all the figures the lead time distributions **Exp**, **Erl**, and **HExp** are coloured blue, black, and red respectively. Note the following:

- In all five arrival processes, as c_{f_1} and c_{f_2} increase simultaneously, the total expected cost rate increases. But the increasing rate for c_{f_2} is high compared with c_{f_1} .
- The Erlang lead time process is associated with a low total expected cost rate, and for the hyper exponential lead time process case the total expected cost rate is high.

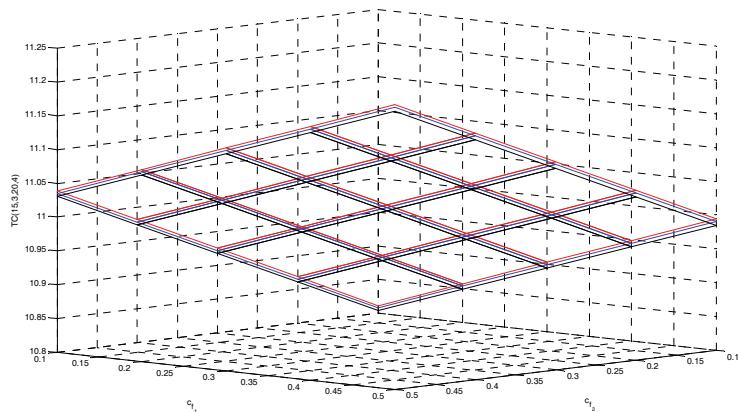


Figure 2: Exp demand process

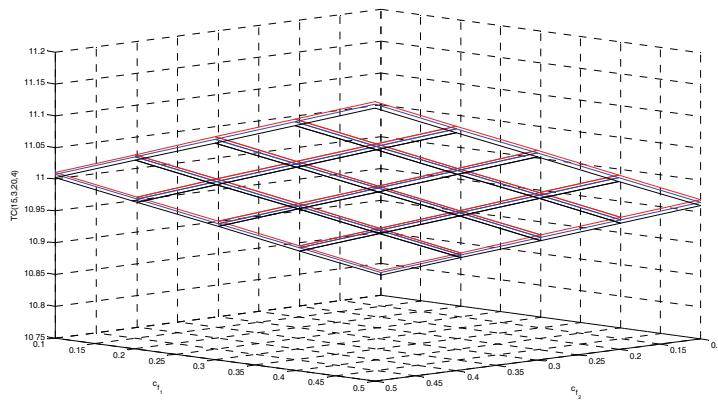


Figure 3: Erl demand process

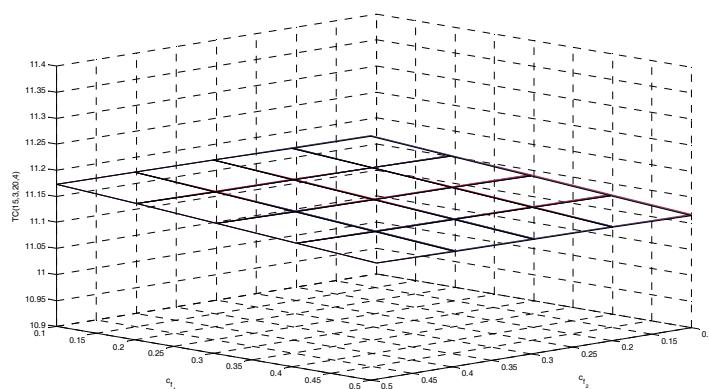


Figure 4: HExp demand process

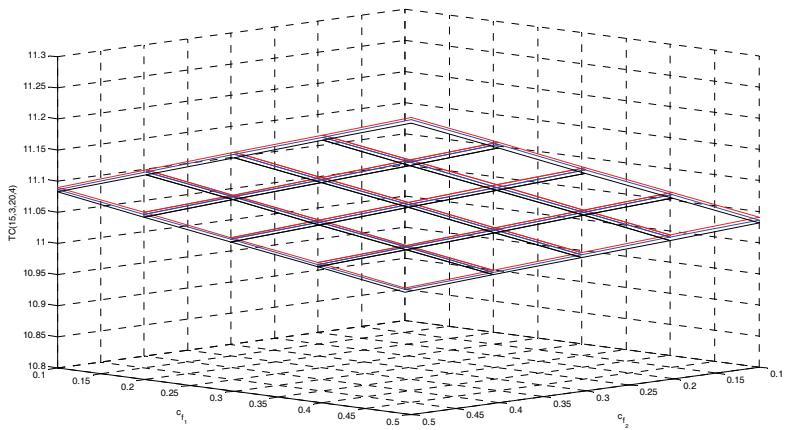


Figure 5: MNC demand process

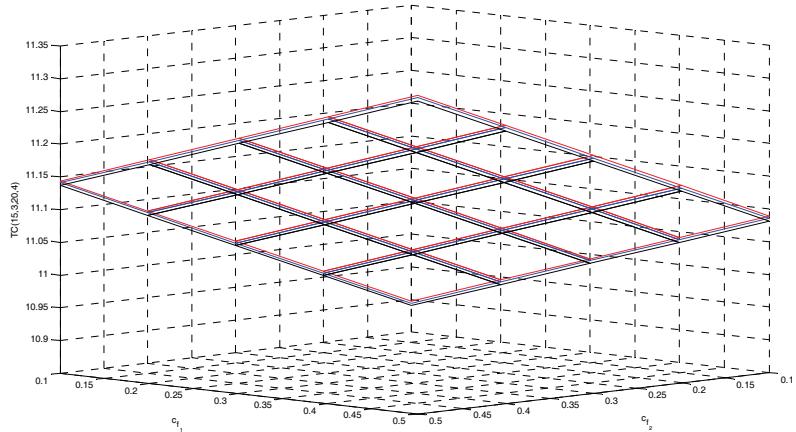


Figure 6: MPC demand process

Example 4: In the final example, the impact of c_{h_1} and c_{h_2} on the total expected cost rate is shown. Consider the following values for the parameters and costs:

$$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \lambda = 15, \beta = 2, \gamma_1 = 0.8, \gamma_2 = 0.4, p_i = 0.6 * 0.4^{i-1}, i = 1, 2, \dots, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$$

The graphs of the total expected cost rate as a function of c_{h_1} and c_{h_2} for the three lead time processes and the five demand processes are shown in Figures 7 to 11. In all the figures the plots for the lead time distributions Exp, Erl, and HExp are coloured blue, black, and red respectively. The following may be observed:

- In all five arrival processes, as c_{h_1} and c_{h_2} increase, the total expected cost rate increases. But the increasing rate for c_{h_2} is high compared with that of c_{h_1} .
- For all the demand processes, the Erlang lead time process has a low total expected cost rate, and a hyper exponential lead time process has a high total expected cost rate.
- The difference between the total expected cost rates for any two lead time processes is high, except for the HExp demand process. For the HExp demand process, the difference between the total expected cost rates for any two lead time processes is low.

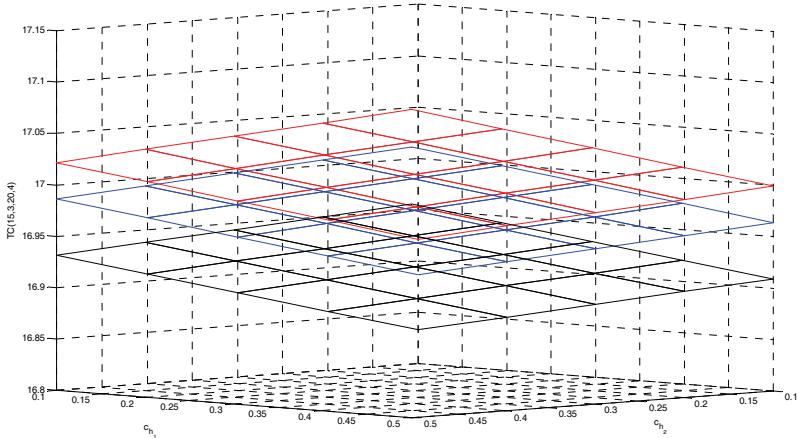


Figure 7: Exp demand process

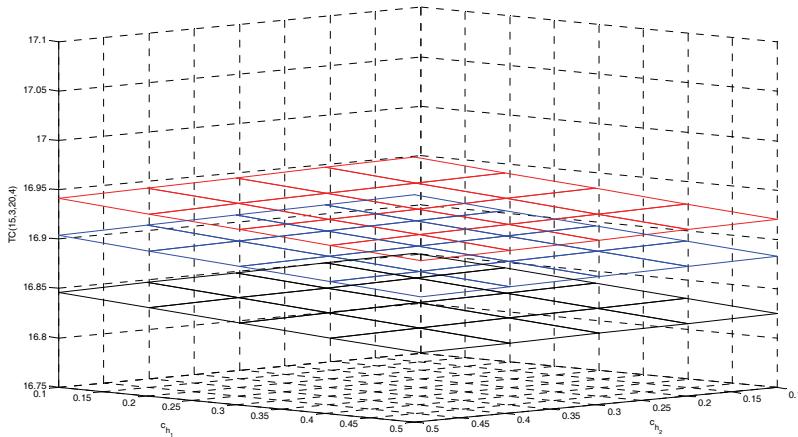


Figure 8: Erl demand process

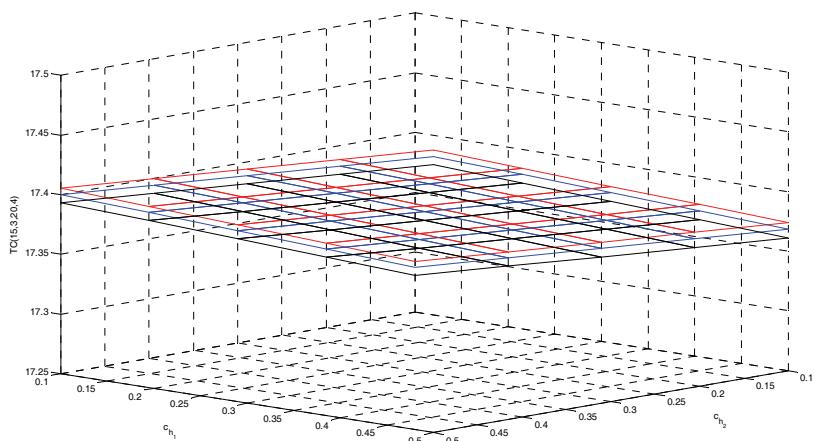


Figure 9: HExp demand process

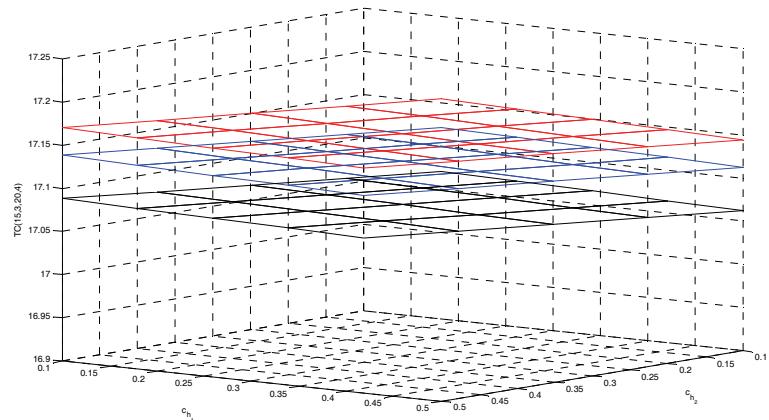


Figure 10: MNC demand process

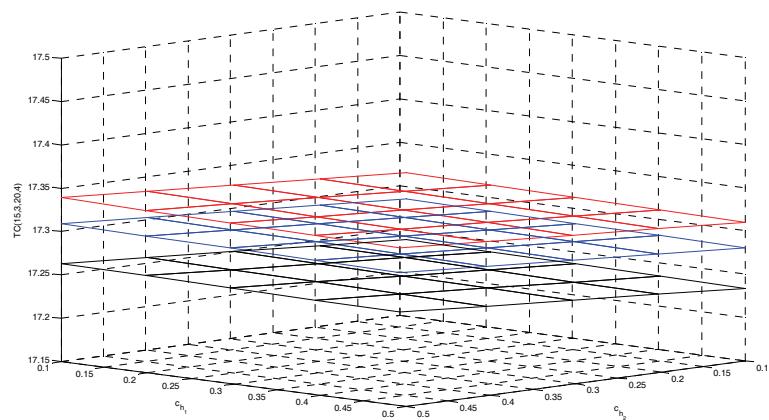


Figure 11: MPC demand process

7. CONCLUSION

This research has extended the existing work on two-commodity continuous review inventory systems by introducing perishability for both commodities, Markov arrival processes for demand time points, and a phase type distribution for lead time. It is also assumed that one of the commodities may accept bulk demands. Steady state solutions are also provided for the joint distribution of inventory levels. For a suitable cost structure, the total expected cost rate in the steady state has been determined. To demonstrate the computability of results derived here, ample numerical illustrations are given. Numerical analysis of the effect of the parameters and costs on the total expected cost rate is given.

8. ACKNOWLEDGEMENTS

V.S.S. Yadavalli would like to thank the National Research Foundation (NRF) for their financial support. G. Arivarignan would like to thank the CSIR - India, for their financial support (No. 25 (156)/07/EMR-II).

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