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An APOS analysis of natural science students' understanding of derivatives

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This article reports on a study which used the APOS (action-process-object-schema) theoretical framework to investigate university students' understanding of derivatives and their applications. Research was done at the Westville Campus of the University of KwaZulu-Natal in South Africa. The relevant rules for finding derivatives and their applications were taught to undergraduate science students. This paper reports on the analysis of students' responses to six types of questions on derivatives and their applications. The findings of this study suggest that those students had difficulty in applying the rules for derivatives and this was possibly the result of many students not having appropriate mental structures at the process, object and schema levels.

Keywords: action, APOS, chain rule, derivatives, process, object, schema, teaching

Introduction

In South Africa, Grade 12 learners are exposed to the concept of derivative of a function $f(x)$, denoted by $f'(x)$. They are exposed tothe following two interpretations of the derivative of $f(x)$: $f'(x)$ gives (1) the gradient of the tangent to the curve f at any point $(x, f(x))$, and (2) the rate of change of f with respect to x. Basic rules to find derivatives of simple functions, mainly polynomial functions and applications of derivatives are also covered. The rules covered are for the basic structures of functions of the types $f(x) = k,\ g(x) = x^n, h(x) = kx^n$ and the sum or difference of such functions (Department of Education, 2003; Department of Education, 2012). The applications of the derivative concept include finding the derivatives of such functions, curve sketching and interpretation of graphs in the context of cubic polynomials, extreme value problems and rates of change. At university level the applications are extended to new and more complex functions, including those based on the concept of composite functions. The findings of other studies indicated that (a) the derivative is a difficult concept for many students to understand (Orton, 1983; Uygur & Özdaş, 2005), (b) students' difficulties with the derivative increase and get worse when the function considered is a composite function (Tall, 1993), and (c) this results in the chain rule being one of the hardest ideas to convey to students in calculus (Gordon, 2005; Uygur & Özdaş, 2007). My interactions with first year mathematics students, at the University of KwaZulu-Natal, indicated that many students experience difficulties in finding the derivatives of functions whose structures include composition of functions. When finding the derivative of a function, for example $f(x) = \ln(3x^2 + 1)$, many students typically respond with $f'(x) = \frac{1}{3x^2+1}$. These indicated that there was a need to engage with a study on students' understanding of the symbolic representations of functions and their derivatives. The research question for this study was: How should the teaching of the concept of the derivative be approached?

Theoretical framework

This study is based on APOS Theory (Dubinsky & McDonald, 2001). APOS Theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept. The mental structures refer to the likely actions, processes, objects and schema required to learn the concept. Research based on this theory requires that for a given concept the likely mental structures need to be detected, and then suitable learning activities should be designed to support the construction of those mental structures.

Asiala, Brown, De Vries, Dubinsky, Mathews & Thomas (1996) proposed a specific framework for APOS Theory based research and curriculum development, in undergraduate mathematics education. The framework consists of the following three components: theoretical analysis, instructional treatment, and observations and assessment of student learning. According to Asiala *et al.* (1996), APOS Theory based research should be done according to the paradigm illustrated in Figure 1.

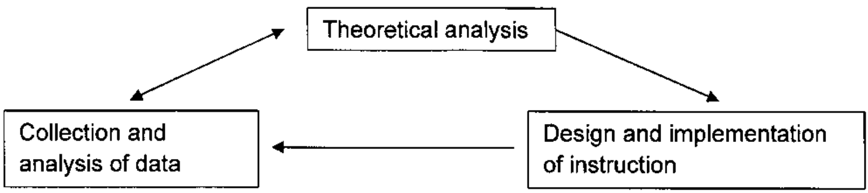


Figure 1 Paradigm: General Research Programme

In this paradigm, theoretical analysis occurs relative to the researchers' knowledge of the concept in question and knowledge of APOS Theory. This theoretical analysis helps to predict the mental structures that are required to learn the concept. For a given mathematical concept, the theoretical analysis informs the design and implementation of instruction. These are then used for collection and analysis of data. The theoretical analysis guides the latter, which Figure 1 indicates could lead to a modification of the initial theoretical analysis of the given mathematical concept.

Description of the APOS/ACE Instructional Treatment

APOS Theory and its application to teaching practice are based on an assumption on mathematical knowledge and a hypothesis on learning. These were developed to understand the ideas of Jean Piaget. In some studies (for example, Weller, Clark, Dubinsky, Loch, McDonald & Merkovsky, 2003), these ideas were recast and applied to various topics in post-secondary mathematics. Piaget investigated the thinking of adolescents and adults, including research mathematicians. Those investigations led him to discover common characteristics, specifically certain mental structures and mechanisms that guide concept acquisition (Piaget, 1970). According to Dubinsky (2010), APOS theory and its application to teaching practice are based on the following assumption on mathematical knowledge and hypothesis on learning mathematics.

Assumption on mathematical knowledge: An individual's mathematical knowledge is his/her tendency to respond to perceived mathematical problem situations and their solutions by [a] reflecting on them in a social context, and [b] constructing or reconstructing mental structures to use in dealing with the situations.

Hypothesis on learning: An individual does not learn mathematical concepts directly. He/she applies mental structures to make sense of a concept (Piaget, 1964). Learning is facilitated if the individual possesses mental structures appropriate for a given mathematical concept. If appropriate mental structures are not present, then learning the concept is almost impossible.

The above imply that the goal for teaching should consist of strategies for: [a] helping students build appropriate mental structures, and [b] guiding them to apply these structures to construct their understanding of mathematical concepts. In APOS Theory, the mental structures are actions, processes, objects and schemas. In the following subsections (1) each of these is briefly described, and (2) then the ACE Teaching Cycle; which constitutes the pedagogical strategies used, based on the hypothesis and the implication for teaching; is described.

After these general considerations, the assumption on mathematical knowledge is focused on by making an APOS analysis of the derivative concept. The result of this analysis is called a *genetic decomposition*. A genetic decomposition of a concept is a structured set of mental constructs which might describe how the concept can develop in the mind of an individual (Asiala *et al.*, 1996). So, a genetic decomposition postulates the particular actions, processes and objects that play a role in the construction of a mental schema for dealing with a given mathematical situation. The *genetic decomposition* arrived at for the derivative concept, is indicated in the methodology section.

APOS Theory

The main mental mechanisms for building the mental structures of action, process, object and schema are called *interiorization* and *encapsulation* (Dubinsky, 2010; Weller *et al.*, 2003). The mental structures of action, process, object and schema constitute the acronym APOS. APOS theory postulates that a mathematical concept develops as one tries to transform existing physical or mental objects. The descriptions of action, process, object and schema; given below; are based on those given by

Weller, Arnon & Dubinsky (2009). I formulated suitable examples to clarify those descriptions.

- Action: A transformation is first conceived as an *action*, when it is a reaction to stimuli which an individual perceives as external. It requires specific instructions, and the need to perform each step of the transformation explicitly. For example, a student who requires an explicit expression to think about the derivative of a function, $f^{'}(x)$, where say $f(x) = x^3$ and can do little more than perform the action $f^{'}(x) = 3x^2$, is considered to have an action understanding of the derivative of a function.
- Process: As an individual repeats and reflects on an action, it may be *interiorized* into a mental *process*. A process is a mental structure that performs the same operation as the action, but wholly in the mind of the individual. Specifically, the individual can imagine performing the transformation without having to execute each step explicitly. For example, an individual with a process understanding of the derivative of a function, say $g(x) = (x^2 + 1)^2$, will construct a mental process which could include that $g(x)$should first be written in a simplified form by squaring the binomial $(x^2 + 1)$and then the derivative can be found by applying the rule, the derivative of a sum of functions is the sum of the individual derivatives of functions.
- Object: If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has *encapsulated* the process into a cognitive *object*. For example, when finding the derivative of functions an individual may confront situations requiring him/her to apply various actions and/or processes. These could include thinking about a function as the composite of two functions, for example the function $h(x) = (x^2 + 1)^{100}$ is the composite of the functions $f(x) = x^{100}$ and $g(x) = x^2 + 1$, since $h(x) = f(g(x))$. To find the derivative, $h(x)$ should first be conceptualized as an object which comprises of the composite of two functions. To this function object, the process understanding for finding derivatives must be encapsulated in the context of the chain rule to find the derivative $h'(x)$.
- Schema: A mathematical topic often involves many actions, processes and objects that need to be organized and linked into a coherent framework, called a *schema*. It is coherent in that it provides an individual with a way of deciding, when presented with a particular mathematical situation, whether the schema applies. For example, the coherence might lie in understanding that to determine the local extrema of a function, say $h(x) = (x^2 - 1)^{100}$, the following must be considered: the derivative $h^{'}(x)$, the critical points of h occur where $h^{'}(x) = 0$, these critical points should be used to construct the

sign diagram of $h^{'}(x)$, and this should be analysed to determine the nature of the extrema of h.

Explanations offered by an APOS analysis are limited to descriptions of the thinking which an individual *might* be capable. So such analyses don't describe what "really" happens in an individual's mind, since this is probably unknowable. Further, the fact that an individual possesses a certain mental structure does not mean that he or she will necessarily apply it in a given situation. This depends on other factors, for example managerial strategies, flexibility, prompts and emotional states. The main use of an APOS analysis is to point to possible pedagogical strategies. Data is collected to validate the analysis or to indicate that it should be reconsidered. For more details, see Asiala *et al*. (1996) and Dubinsky & McDonald (2001).

The ACE Teaching Cycle

This pedagogical approach, based on APOS Theory and the hypothesis on learning and teaching, is a repeated cycle consisting of three components: (A) activities, (C) classroom discussion, and (E) exercises done outside of class (Asiala *et al.*, 1996). The activities, which form the first step of the cycle, are designed to foster the students' development of the mental structures called for by an APOS analysis. In the classroom the teacher guides the students to reflect on the activities and its relation to the mathematical concepts being studied. Students do this by performing mathematical tasks. They discuss their results and listen to explanations, by fellow students or the teacher, of the mathematical meanings of what they are working on. The homework exercises are fairly standard problems. They reinforce the knowledge obtained in the activities and classroom discussions. Students apply this knowledge to solve standard problems related to the topic being studied. The implementation of this approach and its effectiveness in helping students make mental constructions and learn mathematics has been reported in several research studies. A summary of early work can be found in Weller *et al*. (2003).

Insights on the Derivative Concept from Past Studies

In any first course on calculus, the building block for the concept of a derivative is the function concept. For this reason the literature review focuses on (1) Functions and function composition, and (2) Derivatives, including the chain rule.

Functions and function composition.

Functions are a central part of the pre-calculus and calculus curriculum (Tall, 1997). This was supported by a study of first year calculus prescribed textbooks (for example the textbooks by Stewart, 2009; Lial, Greenwell & Ritchley, 2008) at the University of KwaZulu-Natal, which revealed that a good grounding of the following are pre-requisites for introducing the derivative concept: [a] algebraic manipulations, and [b] functions, including their symbolic and graphical representations. These imply that before the concept of the derivative of a function is introduced, students should have adequately established algebraic manipulation skills, and the concept of a function. Their mental structures of functions should be at the higher levels of APOS; process level and higher. The reality is that at best most students are at the process level, since in general students exhibit a predominant reliance on the use of and the need for algebraic formulae when dealing with the function concept (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Asiala, Cottrill, Dubinsky & Schwingerdorf, 1997; Akkoç & Tall, 2005). Further, previous studies emphasized the importance of the concept of function composition in the understanding of the chain rule (Clark, Cordero, Cottrill, Czarnocha, De Vries, St. John, Tolias & Vidakovi´c, 1997; Cottrill, 1999; Hassani, 1998), which is a technique used to find the derivative of functions, in whose structures other functions are embedded; for example $f(x) = \sqrt{x^2 + 1}$. Here, we can conceptualize $f(x)$ as a composition of two functions, for example $f(x) = g(h(x))$ where $g(x) = \sqrt{x}$ and $h(x) = x^2 + 1$.

Derivatives, including the chain rule.

The derivative is a difficult concept for many students to understand (Orton, 1983; Uygur & Özdaş, 2005). Giraldo, Carvalho & Tall (2003) distinguish between a *description* of a concept, which specifies some properties of that concept and the formal concept *definition*. They noted that a commonly used description of the derivative is the following: gradient of a function $f(x)$ at x_0 is the slope of the tangent line to the curve f at the point $(x_0, f(x_0))$. Hähkiöniemi (2004) noted that students' understanding of the derivative can be improved if they are exposed to several different kinds of representations, to process the derivative. He exposed students to different perceptual (eg. rate of change of the function from the graph, steepness of tangent) and symbolic (eg. differentiation rule, slope of a tangent) representations of the concept of a derivative. Perceptual representations assist students to understand the derivative as an object. Roorda, Vos & Goedhart (2009) focused on representations and their connections as part of understanding derivatives. They found that growth in understanding depends on a variety of connections, both between and within representations, and also between a physical application and mathematical representations. Zandieh (2000) observed that students prefer the graphical representation in tasks and explanations about derivatives. This was supported by Tall (2010) who made a strong argument for direct links between visualization and symbolization when teaching the concept of a derivative.

When the function considered is a composite function, students' difficulties with the derivative increase and get worse (Tall, 1993). This results in the chain rule being one of the hardest ideas to convey to students in calculus (Gordon, 2005, Uygur & Özdaş, 2007). Clark *et al.* (1997) who studied students' understanding of the chain rule and its applications concluded that the difficulties with the chain rule for a large number of students could be attributed to student difficulties in dealing with composition and decomposition of functions. The implication is that the understanding of composition of functions is an integral part to understanding the chain rule, which is supported by several studies (eg. Webster, 1978; Cottrill, 1999; Horvath, 2007).

It seems that many students perform poorly because they: (a) are unable to adequately handle information given in symbolic form which represent objects [abstract entities], for example functions, and (b) lack adequate schema or frameworks, which help to organize and link different objects (Maharaj, 2005). The teaching implication is that (1) a variety of representations should be used, and (2) students should be encouraged to engage with a flexibility of mathematical conceptions (Andresen, 2007; Maharaj, 2010) of $f'(x)$, the derivative of the function f with respect to x.

Participants and Methodology

This section focuses on (1) Participants, aim of module studied and ethical issues, (2) Theoretical analysis of derivative concept using an APOS approach, genetic decomposition, (3) Instructional treatment, ACE teaching cycle, and (4) Tools for collection of data.

Participants, Aim of Module Studied and Ethical Issues

The participants in this study were 857 science students at the University of KwaZulu-Natal (UKZN) in 2010; about 67% of these were first year students and about 33% were senior (for example, second or third year) students. Those 857 students took the third test for a module they were studying, and it was convenient to analyse their results, so it was a convenience sampling. That test was comprised of multiple-choice-questions; it was neither a standardised test, nor was it adapted from literature. The students were studying the Math133W1 (Mathematics & Statistics for Natural Sciences) module, which was a compulsory service module towards their Bachelor of Science degrees. Major subjects for these students varied over chemistry, physics, biology, zoology and pharmacy. The aim of that module studied was to "introduce students to the fundamental principles, methods, procedures and techniques of mathematics and statistics as the language of science" (Faculty of Science and Agriculture, 2010:229). Those students attended their lectures in one of three timetable groups. The study was conducted according to the research ethical guidelines of UKZN as indicated in Research Policy V (University of Kwa-Zulu Natal, 2007). Also, the researcher successfully completed the United States of America's National Institutes of Health (NIH) Office of Extramural Research Web-based training course "Protecting Human Research Participants" and was certified by that NIH. Guidelines indicated in that course were also followed in this study. This was the context for the theoretical analysis of the derivative concept.

Theoretical Analysis of Derivative Concept using an APOS Approach, Genetic Decomposition

My theoretical analysis indicated the type of mental structures of action, process, object, and schema relevant to both the derivative concept and types of problems based on the derivative that the participants encountered. Those mental structures were described under the subheading *APOS Theory*. The examples I gave there to expand on the descriptions of the mental constructions, and my literature review (Tall, 1997; Clark *et al.*, 1997; Cottrill, 1999; Hassani, 1998) led me to the following genetic decomposition.

As part of his or her function schema, the student has developed

1. a process or object conception of a function, and
2. a process or object conception of composition of functions.

As part of his or her derivative schema, the student has

3. an action conception which enables the finding of derivatives of simple functions, whose rules are given in the symbolic form. For example, $f(x) = 3x^2$.
4. a process conception of differentiation which enables the finding of derivatives of functions. This could involve studying the structure of the function, detecting whether a rule for differentiation could be applied or whether the function should be written in a standard form which enables the application of the appropriate rules for differentiating.
5. an object conception which enables the seeing of strings of processes as a totality and performing mental or written actions on the internal structure of the given function, which enables differentiation. For example, the student views a function $h(x)$ as an object which is a composition of two functions, $h(x) = f(g(x))$, to which the chain rule can be applied.
6. organized the actions, processes and objects related to the derivative concept, and linked them into a coherent framework. This framework includes various interpretations of the derivative in different contexts, and possible techniques for [a] finding derivatives of various function types, [b] interpreting the graph of a derivative, [c] using the derivative of a function for curve sketching, or [d] optimizing a function.

Points 1 and 2 of my genetic decomposition; dealing with the function schema; is the same as that of Clark *et al.* (1997). However, the points dealing with the derivative schema are different. The reason for the latter is that the genetic decomposition given by Clark *et al.* (1997) was for the chain rule, and not for the derivative (in general) and related applications.

Instructional Treatment, ACE Teaching Cycle

This was informed by my theoretical analysis of the derivative concept, the literature review (Asiala *et al.*, 1997; Akkoç & Tall, 2005; Hähkiöniemi, 2004) and the types of problems that the students had to be exposed to. A lecture schedule, based on sections of the prescribed book, was followed. That schedule allocated 22 lecture periods, each of 45 minutes' duration, for sections and topics considered to be crucial for the derivative concept as determined by my genetic decomposition of the derivative concept, and informed by the literature review (Hähkiöniemi, 2004; Roorda *et al.*, 2009; Tall, 2010). There were 4 periods per week and a 3 hour tutorial session. During the lecture periods, students were exposed to: (a) algebraic manipulation skills, and symbolic and graphical representations of functions. (b) the two interpretations of derivative of a function which are dealt with in grade 12, at schools. (c) basic differentiation techniques for different types of simple functions, including for example $y = e^{kx}$, $y = a^{kx}$, $y = \log_a x$. (d) the composition and decomposition of functions, followed by applications of the chain rule. (e) applications of the derivative to curve sketching, and optimization problems in their fields of study. For each of the lecture periods, activities were formulated and were projected by use of a PC tablet.

I now focus on how the teaching and learning experience was structured for students, by using an ACE teaching cycle approach. For example, one of those periods dealt with the question: How to determine where a function is increasing or decreasing? Appendix A gives activities for this question. A reasonable time was given for students to reflect and work on each activity; they were free to discuss with other students sitting beside them and to use the prescribed textbook. While students engaged with the activities I observed how they worked, and noted their difficulties and aspects that required further explanations. These informed my explanations; using the PC tablet; to the class. Activities and classroom discussions were followed by homework exercises, which students had to work on as part of their tutorial requirements. The PC Tablet was used to summarize the lecture-room discussions. Those summaries were available to students, on the website for the Math133W1 module. During each of the six two-and-a-half hour tutorial sessions, students were in groups of about 35. In their groups they could further discuss the homework exercises with their tutors.

Tools for Collection of Data

About a week after the last tutorial on derivatives and their applications, a multiple choice test was administered to 857 students. The questions were set by me and were similar to those for the activities and homework. Students were required to first work out the solutions in the space below each question and then to mark their choices on the MCQ (multiple-choice-question) cards. The MCQ test was not a standardised test adapted from literature. Each question was given a weighting of 3 marks. Students were informed that to discourage guessing, negative marking applied; -1 for each incorrect choice. In 2010 at UKZN, negative marking was a science faculty requirement for the use of MCQs. The six questions are indicated in the next section. Those questions were constructed by using my genetic decomposition and notes on relevant observations during student activities and tutorials. The options given for each of the MCQs were constructed bearing in mind the APOS levels of mental structures.

Analysis, Findings and Discussion

To represent the analysis, findings and discussion for each of the six questions in a reader friendly format, the following subheadings which describe the type of question are used: (1) Decomposition of functions, (2) Derivative of an exponential function with base e, (3) Derivatives of exponential and logarithmic functions, (4) Rates of change, (5) Interpretation of the graph of a derivative, and (6) Optimization of a function. Under each of these subheadings the relevant test item and question analysis is given. The question analysis indicating student responses for each of the six questions are indicated in Tables 1 to 6. In each of these tables 'Qu' means question and the symbol "*" appears under the letter of the correct answer, for example B is the correct answer for question 1. The Bad Index gives the number of students who marked more than one choice.The numbers under the columns A to Omit Index gives the percentage of students, rounded correct to two decimal places.

Decomposition of Functions

1. If the function defined by $y = \frac{1}{x^2-7}$ is expressed as the composition of two functions f and g such that $y = f[g(x)]$ then

A) $f(x) = \frac{1}{x^2}$; $g(x) = x - 7$ B) $f(x) = \frac{1}{x}$; $g(x) = x^2 - 7$

C) $f(x) = \frac{1}{x^2}$; $g(x) = -\frac{1}{7}$ D) $f(x) = \frac{1}{7}$; $g(x) = x^2 - 7$

E) none of these

Table 1 Student responses in percentages (n = 857)

Qu	A	B	C	D	E	Omit Index	Bad Index
1	4.32	67.21*	10.74	0.01	2.33	14.47	1

Question 1 is based on recognizing a given function as the composition of two functions, which is the reversal of the process of composing two functions (which coincides with point 2 of my initial genetic decomposition). Table 1 suggests that [a] about 67.21% of the students had a function schema which included an adequate process or object conception of composition of functions, and [b] at least 17.4% of the students (using the results of the four other choices) had mental structures for the composition which were, at best at the action level. If this is accepted then at least 17.4% of the students had inadequate schema for composition of functions, and this should impact negatively on their application of the chain rule. For the 14.47% of students who gave no response, there could be a number of reasons for this. For example reasons could be that: (1) some of them had inadequate composition of function schema which could be a result of students' difficulties in dealing with the composition and decompositions of functions (Clark *et al.*, 1997), or (2) some avoided indicating an uncertain choice because of negative marking. The literature review (Webster, 1978; Cottrill, 1999; Horvath, 2007) suggested that an insight of composition of functions is required for success with the chain rule for differentiation. So, one would expect about 32% of these students to have difficulties with the chain rule (see items 2 and 3, Tables 2 and 3) since function composition increases students' difficulties with the derivative (Tall, 1993) and hence the chain rule (Gordon, 2005, Uygur & Özdaş, 2007). Note that item 1 was designed to assess point 2 of my genetic decomposition which is a pre-requisite for the description given in point 6, of that genetic decomposition.

Derivative of an Exponential Function with base *e*

2. If $y = e^{4x^2} + x$ then $\frac{dy}{dx} =$

A) $4x^2e^{4x^2-1} + 1$ B) $8xe^{2x} + 1$ C) $8xe^{x^2} + 1$
D) $8xe^{4x^2} + 1$ E) $e^{8x} + 1$

Table 2 Student responses in percentages (n = 857)

Qu	A	B	C	D	E	Omit Index	Bad Index
2	8.57	1.63	1.75	81.45*	2.45	3.97	0

The function in Question 2 has the structure of the sum of two functions. It is based on recognizing the rules for differentiation that should be applied are for the sum of functions and the chain rule to an exponential function, with base *e*. In terms of my genetic decomposition, this question requires a derivative schema which coincides with the descriptions indicated under points 4 and 5. Table 2 suggests that 81.45% of the students had an adequate derivative schema for functions of this type, and 14.4% had schema which excluded the chain rule. Since 81.45% had a schema which included the chain rule as a rule for differentiation, this suggests that a schema for decomposition of functions as suggested by Clark *et al.* (1997), 67.21% from Table 1, may not be conclusively the determining factor for application of the chain rule. This could be viewed as a new finding and seems reasonable, since an application of the chain rule only requires that one recognizes [a] that this rule should be applied, and [b] which is/are the embedded function(s). It does not actually require the decomposing of the original function into sub-functions, as focused on in item 1.

Derivatives of Exponential and Logarithmic Functions

3. If $f(x) = (23)^{-x} + \ln(9+x^2)$ then $f'(x) =$

A) $-\ln 23(23)^{-x} + \frac{1}{9+x^2}$ B) $\ln 23(23)^{-x} + \frac{2x}{9+x^2}$

C) $-x(23)^{-x-1} + \ln(9) + \ln(2x)$ D) $(23)^{-x} + \frac{2x}{9+2x}$

E) $-\ln 23(23)^{-x} + \frac{2x}{9+x^2}$

Table 3 Student responses in percentages (*n* = 857)

Qu	A	B	C	D	E	Omit Index	Bad Index
3	5.13	21.12	12.95	6.53	42.24*	12.02	0

The function structure is the sum of an exponential function, with base not e, and a natural logarithmic function. Each of those functions requires the application of the chain rule. In terms of my genetic decomposition, the successful answering of this type of question requires a derivative schema which coincides with the descriptions indicated under points 4 and 5. According to my genetic decomposition, students' responses in Table 3 suggest: [a] 12.95% (from response C) were at best at the action level, [b] 26.25% (from responses for A and B) were at a process level of the chain rule applied to only one of those functions, [c] 6.53% had no adequate schema for differentiating an exponential function of the indicated type, and [d] 42.24% of students had an adequate schema. So, about 67% of those students had inadequate schema to find derivatives of exponential and logarithmic functions which required two separate applications of the chain rule, relating to point 6[a] of my genetic decomposition. This seems to support literature findings (Gordon, 2005, Uygur & Özdaş, 2007) that the chain rule is one of the hardest ideas to convey to students in calculus.

Rates of Change

4. The population of a certain type of insect in a region near the equator is given by $P(t) = 15\ln(t + 10)$, where t represents the time in days. The rate of change of the population when $= 2$, is

A) 2.5 insects B) 3.4 insects C) 1.75 insects

D) 1.5 insects E) 1.25 insects

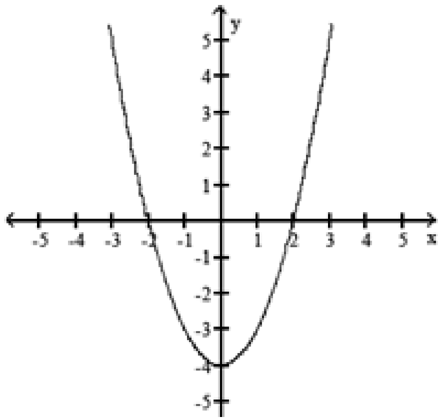
Table 4 Student responses in percentages (*n* = 857)

Qu	A	B	C	D	E	Omit Index	Bad Index
4	8.40	3.03	1.75	2.68	68.26*	15.75	1

Question 4 required interpreting [a] firstly as a derivative problem, and [b] the derivative as a rate of change. The successful answering of such a question requires a derivative schema incorporating a framework as described under point 6 of my genetic decomposition, including a flexibility of conception of the derivative concept. Student responses in Table 4 suggest that 68.26% had such a schema, enabling them to successfully interpret the derivative as a rate of change. This could be a result of the teaching approach which focused on several different kinds of representations of the derivative (Hähkiöniemi, 2004) including explanations on the direct links between visualization and symbolization of the derivative concept (Tall, 2010).

Interpretation of the graph of a derivative

5. Consider the graph



If the graph is that of $f'(x)$, then the function f has

A) relative maxima at -2 and 2
B) a relative minimum at -4
C) a relative maximum at -2; a relative minimum at 2
D) a relative minimum at -2; a relative maximum at 2
E) critical values at $-2, -4$ and 2

Table 5 Student responses in percentages (n = 857)

Qu	A	B	C	D	E	Omit Index	Bad Index
5	5.25	35.24	22.29*	11.79	12.02	13.42	0

Question 5 required an analysis of the graph of the derivative function $f'(x)$, to determine the intervals over which the original function $f(x)$ is increasing or decreasing and the use of this information to find the relative extrema of $f(x)$. According to my genetic decomposition of the derivative schema, the successful answering of such a question requires the description given under point 6. In particular this assessment item requires that the student interprets the derivative represented in graphical form. Interpretation is required at a level that should enable the making of relevant conclusions relating to the *original function* $f(x)$ on: (1) the intervals over which $f(x)$ increases or decreases, and (2) the relative extrema of $f(x)$. The student responses in Table 5 suggests that [a] only 22.29% had an adequate schema which enabled them to interpret the graph of the derivative function at the required level, and [b] 53.51% (from responses for A, B and E) had mental structures which were not even at an action level, for what was required. The latter suggests that there should be greater emphasis during teaching on derivatives represented graphically (Zandieh, 2000) and relevant connections (Roorda *et al.*, 2009) on where the original function is increasing or decreasing.

Optimization of a Function

6. The percent of concentration of a certain drug in the bloodstream x hours after the drug is administered is given by $K(x) = \frac{3x}{x^2+4}$, and $K'(x) = \frac{-3x^2+12}{(x^2+4)^2}$.
The time (in hours) at which the concentration is a maximum, and the maximum concentration, are respectively given by
A) $1; \frac{3}{5}\%$ B) $0.5; \frac{1}{3}\%$ C) $2; \frac{3}{4}\%$ D) $3; 1\%$ E) $-2; -\frac{3}{4}\%$

Table 6 Student responses in percentages (n = 857)

Qu	A	B	C	D	E	Omit Index	Bad Index
6	6.88	6.07	46.09*	3.62	8.63	28.70	0

Question 6 required the optimization of a function in a practical context. A successful answering of such a question would require that a student has a schema which includes a description of what is indicated under point 6 of my genetic decomposition. Table 6 suggests that 54.72% of students (from responses to C and E) had some sort of schema for optimization of a function, of those 46.09% had an adequate schema for optimizing a function in a *practical context*, while 8.63% did not. The teaching implication here is that growth in understanding derivatives depends on a variety of connections, in this case between a physical application and a mathematical representation (Roorda *et al.*, 2009).

Conclusions and Recommendations

Useful insights into the relevant mental structures towards which teaching should focus were revealed by the APOS genetic decomposition of the derivative concept and its applications. This helped to design learning activities for the derivative concept, some of which were fairly successful. The findings of this study suggest that students seem to experience difficulties when: (1) differentiating a function which requires the application of the chain rule, and (2) interpreting the derivative of a function represented in graphical form. For (1) it seems that composition of functions positively influence the application of the chain rule, so this concept should preferably be focused on just before the chain rule is introduced as a differentiation technique. In particular the detecting of which functions are involved in the composition of a given function could aid the application of the chain rule. However, it seems that more emphasis should be placed on the detecting of embedded functions, to which the chain rule should be applied. This implies that point 2 of my genetic decomposition for the function schema should be modified to focus on which sub-function(s) is/are imbedded within a function, in particular process or object conceptions of detecting embedded functions (when applying the chain rule). For (2) it seems there is a need to help students set up an appropriate schema. This could include unpacking the information on the derivative represented in graphical form to a table of signs representation for $f'(x)$. For question 5, the table of signs for $f'(x)$ is given in Figure 2. This table could then be used to analyse the behaviour of the function $f(x)$ over the relevant intervals, and also to find relative extrema.

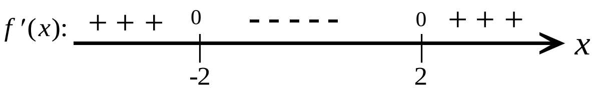


Figure 2 Table of signs for $f'(x)$

My reflections on the instructional design indicated that more time needs to be devoted to helping students develop the mental structures at the process, object and schema levels. This implies that instruction should focus on (1) verbal and graphical approaches to applications on the derivative concept, (2) unpacking of structures given in symbolic form and information given in graphical form, and (3) modelling possible schema. A graphical approach should facilitate the development of mental structures at the process and object levels, while a focus on symbolic structures should aid object conceptions. If schemas organize and link the relevant actions, processes and objects then this should be a part of the instructional treatment. The impact of such a focus on instruction will require further research. Further, the present study provides six assessment tasks on the derivative concept and applications that were designed within an internationally tested research framework. These could lead to a study on the following research question: *What insights would an APOS analysis of students' understanding of the derivative concept reveal?*

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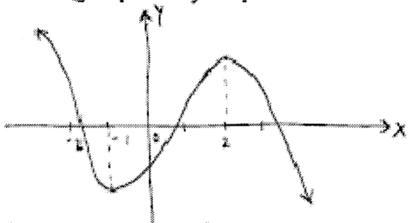
Appendix A An example of an activity

How to determine where a function is increasing or decreasing?

1. Consider the graph of f below:



1.1 Give the intervals on which
a) f is decreasing
b) f is increasing

1.2 On an interval how is the increase or decrease of f related to $f'(x)$?

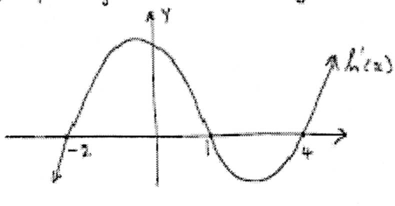
2. What is a critical number of a function?

3. Let $g(x) = x e^{x^2 - 3x}$

3.1 Find the critical numbers of g.

3.2 Find the intervals on which g is increasing or decreasing.

4. The graph of h' is given below:



Find the intervals on which h is increasing or decreasing.