South African Journal of Education
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Vol 28:401-414

Some insights from research literature for teaching and learning mathematics

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I report on the findings from research and literature on (a) use of symbols in mathematics, (b) algebraic/ trigonometric expressions, (c) solving equations, and (d) functions and calculus. From these, some insights and implications for teaching and learning are derived.

Keywords: calculus; equations; expressions; functions; learning; mathematical symbolism; teaching

Introduction

Various studies (e.g. Stacey, 1988; Vinner, 1991; Kieran, 1992; Esty, 1992; Sfard & Linchevski, 1994; Bell, 1995; Linchevski & Herscovics, 1996; McDowell, 1996; Souviney, 1996; Dreyfus, 1999; Lithner, 2000; Mason, 2000, Maharaj, 2005) have focused on the teaching and learning of school mathematics. These studies have indicated some important sources of students' difficulties in mathematics. Kieran (1992) considered a student's inability to acquire an in-depth sense of the structural aspects of algebra to be the main obstacle. Sfard and Linchevski (1994) have analysed the nature and growth of algebraic thinking from an epistemological perspective supported by historical observations. They indicated that the development of algebraic thinking was a sequence of ever more advanced transitions from operational (procedural) to structural outlooks. Mason (2000:97) has argued that "... the style and the nature of questions encountered by students strongly influences the sense that they make of the subject matter". The questions that come to the mind of an educator are influenced by the perspective and disposition that he/she has towards mathematics and pedagogy (Mason, 2000). These questions in turn influence the sense learners make of the subject matter. In this article I focus on the outcomes and implications of research on (a) use of symbols in mathematics, (b) algebraic/trigonometric expressions, (c) solving equations, and (d) functions and calculus.

Dicussion on research findings

Use of symbols in mathematics

Symbols are used in many different contexts in mathematics. This can be to represent: technical concepts (e.g. unknown, coefficient, variable) operations (e.g. $+$, $-$, $\sqrt{}$) and expressions (e.g. $3x + 1$, $2 \sin \theta + 1$) or equations (e.g. $ax^2 + bx + c = 0$, $\cos 2x + \cos x - 2 = 0$). The many uses of literal symbols in algebra have been documented by Philipp (1992). Some of the different uses of symbols in secondary school mathematics are given in Figure 1. Symbols can represent various concepts and also take on varying roles. The context is important in determining the role of the literal symbol (Philipp, 1992). Therefore the correct reading of the context could pose a problem to learners of mathematics. This view is supported by Kieran (1992:396) who argued that discriminating "... the various ways in which letters can be used in algebra can present difficulties to students". Therefore the different notions of letters in the context of algebraic symbolism could imply different levels of difficulty for learners. Algebraic symbolism in its broad use includes symbols in the various sections of school mathematics, for example, algebra, calculus, analytical geometry, geometry and trigonometry. From a cognitive point of view, tasks such as grouping algebraic terms and using algebraic expressions demand "... quite an advanced perception of literal symbols" (Linchevski & Herscovics, 1996:43). Hiebert and Carpenter (1992:72) have suggested

> Once meanings are established for individual symbols, it is possible to think about creating meanings for rules and procedures that govern actions on these symbols.

labels km, m in 1km = 1000m	*generalised number* $3x + 2x = 5x$
constants c in $3x + c$ c in $\sin\theta + c$	*varying quantities* x, y in $y = 3x + 1$ x, y in $y = 3\sin x + 1$
unknowns x in $2x + 1 = 0$ θ in $2\sin\theta + 1 = 0$	*parameters* m, b in $y = mx + b$ a, b in $y = a\sin x + b$
In geometry and trigonometry: *vertices* A, B and C in triangle ABC *shortforms* PQ ∥ MR	 *sides* BC or a in triangle ABC *angles* $\hat{A}_1$, $\hat{A}_2$ or $B\hat{A}C$

Figure 1 The many uses of symbols in mathematics

The teaching implication is that before students are required to use and manipulate algebraic and trigonometric expressions, the meanings of the symbols must be established. This should be done when symbols are introduced in the context of the various topics in different sections of the mathematics syllabus. In mathematics, different classes of symbols can be used to distinguish and [to] reveal the essentially different identities of an object which is being treated in two different ways (Harel & Kaput, 1991). For example, the identity element for addition is denoted by the symbol 0 when working in the real number system, while the symbol $0 + 0i$ is used when working in the complex number system. Harel and Kaput (1991:91) have noted that certain symbols "... include features that reflect the structure of mathematical objects, relations or operations that they stand for". Examples of such elaborated symbols are $(x; y)$ for an ordered pair of numbers, and for a specific real-valued function. Attention must therefore be focused on establishing the meaning of symbols as they appear in different contexts.

Algebraic and trigonometric expressions

Some studies (Sfard & Linchevski, 1994; Kieran, 1992) have provided a detailed analysis of cognitive obstacles as a result of the dual interpretation of algebraic expressions — operational and structural. Linchevski and Herscovics (1996:42) have suggested that research studies show "... simplification of algebraic expressions creates serious difficulties for many students". This means that teachers should acknowledge and appreciate the difficulties that are experienced by learners. These difficulties are related to the deletion error, conventions of algebraic syntax, and the gaps between arithmetic, algebra, and geometry.

Deletion error

Deletion error is illustrated when students simplify an expression, say $9x - 4$ to $5x$, or $9 \tan x - 4$ to $5 \tan x$. Kieran (1992) has indicated that Carry, Lewis and Bernard observed this type of error in a study of the equation-solving processes used by college students. These researchers attributed the deletion error to the over-generalisation (or false generalisation) of certain mathematically valid operations. The source of the deletion error can be traced back to arithmetic, where simplification gives a single numerical value. So it seems that students who provide such answers simplified $9x - 4$ by first 'deleting' x and treating the expression as $9 - 4$, and then tacking on x again. This is supported by Kieran (1992:398) who has suggested that some students tend to "... simplify algebraic expressions by computing according to the rules of arithmetic and then tack on the letters".

Conventions of algebraic syntax

Students often write $3x - 2$ when simplifying, for an example, expressions such as $3(x - 2)$. This can be explained by the fact that beginner algebra students tend to read expressions; as everyone else does when reading sentences in English; from left to right and therefore do not see the need for brackets (Kieran, 1992). Further, they do not read and apply the equal sign as an equivalence relation in the context of equations (Sfard & Linchevski, 1994). Students should be aware of the conventions of algebraic syntax since this gives meaning to algebraic expressions and equations. They should also learn where to use brackets and where not, since bracketing structures the text. Teaching should therefore focus on the conventions of algebraic syntax. This has many advantages for learners of mathematics. In this regard Cangelosi (1996:137) has noted that, provided

> ... a person has been taught the meanings of the symbols and has become accustomed to using them, the compact form with the shorthand notation makes it easier to recognise critical relationships ...

The correct interpretations of these conventions reveal the power of mathematical symbolism. For example, when simplifying the product of two binomials, say, particular attention should be paid to the relationship between coefficients and signs in the factors, and those that occur in the quadratic product (Roebuck, 1997). This promotes the interpretation of the equal sign as an equivalence relation.

Arithmetic and algebra, algebra and geometry: visualisation, language and emergence of symbolism

When introducing algebra the use of letters should be withheld until it is evident that learners are ready for their use, and teaching should recognise and prepare learners for the various uses of letters in algebra as the need arises (Harper, 1987; Stols, 1996). Pyke (2003:406) has shown that the learners' use of "… symbols, words, and diagrams to communicate about their ideas each contribute in different ways to solving tasks". The structurality of geometry and the visual overview that it provides facilitate thinking and effective investigation (Sfard, 1995). For example, the formulae for determining the areas of squares and rectangles can be used to introduce algebraic expressions. Such an approach could help learners to make links between arithmetic and algebra. A teaching sequence which allowed students to develop a procedural (operational) meaning for algebraic expressions such as $4x + 4y$ was designed by Chahouh and Herscovics (Kieran, 1992). The researchers noted that students tended to regard such expressions as incomplete unless they formed part of an equality, for example $4x + 4y$ = Area. This suggests that a procedural conception for algebraic expressions requires in the mind of the student a final result as the end product of the procedure (as in arithmetic). Such reluctance by students to accept a lack of closure should therefore be appreciated by teachers. Kieran (1991:49) described the research findings of a teaching approach developed by Peck and Jencks "... that helps students make explicit links between their arithmetic and the nonnumerical notation of algebra". Students were exposed to an approach for simplifying 24×26 (arithmetic), based on areas of rectangles and squares using the geometric illustration in Figure 2.

This approach enabled students to record statements such as

$$(x + 4)(x + 6) = x^2 + 10x + 24.$$

Peck and Jencks also observed that this approach led students to handle expressions of the type $(x + 9)(x - 4)$ which allowed generalisations such as

$$(a + b)^2 = a^2 + 2ab + b^2$$

to be viewed as simple variations of the same conceptual theme (Kieran, 1991).

Various studies (e.g. Ernest, 1987; Burton, 1988; Pegg & Redden, 1990; Wessels, 1990; Oldfield, 1996) have suggested that there should be a greater focus on language. In support, researchers (e.g. Esty & Teppo, 1996; Usiskin, 1996; Pyke, 2003) have agreed that language plays an important role in the teaching and learning of mathematics, and that this subject makes use of a

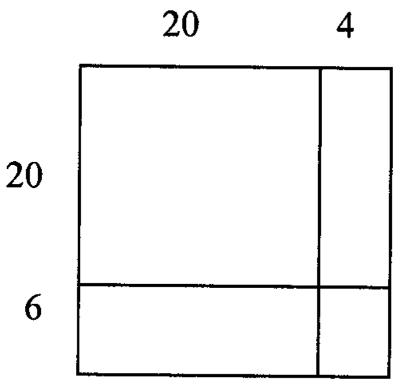

Figure 2 Geometric illustration of 24 × 26

special language. The special language refers to symbolic notation which fills a dual role as an instrument of communication and thought (Roux, 2003). This is what makes it possible to represent mathematical concepts, structures and relationships in symbolic form. Burton (1988:2) has suggested

> ... a major component of student difficulty with algebra is the inability to make sense of the algebraic system as a language, and accordingly that remedies should be sought by considering algebra in a linguistic sense.

Continuing on this theme, Pegg and Redden (1990:19) concurred that an approach to mathematics that is often neglected is the "... role of language as the link between experiences with number patterns and the emergence of algebraic notation". They (Pegg & Redden, 1990:19-21) have suggested an approach to algebra based on

a. Experiencing activities with number patterns.
b. Expressing the rules which govern particular number patterns in English sentences.
c. Writing the rule(s) which govern number patterns in an abbreviated form.

Contributing further to this theme, Bell (1995) described how the generality or non-generality of proposed patterns observed on a calendar page could be used to introduce algebra and appropriate symbolic notation.

Teaching should also emphasise the importance of extracting meaning from information represented in coded form via symbolic notation. For example learners in Grade 11 encounter the sine rule in the form:

In any $\triangle$ *ABC*, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

An appropriate verbalisation of this rule such as, ... *In any triangle the ratio formed by taking the length of a side and dividing by the sine of the angle* *opposite this side is constant* ..., could help learners understand or make sense of the deep ingrained meaning of this rule. Each of the above approaches is aimed at a transition from a procedural (operational) to a structural conception of the relevant concepts in mathematics. Future success in mathematics requires a structural interpretation and the ability to use modern symbolic notation. Both of these are also required when manipulating expressions and solving equations.

Solving equations

Various studies (e.g. Kieran, 1992; Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996) have focused on solving linear equations. For example, Linchevski and Herscovics (1996:44) noted

> ... for a large number of high-school students, there are many cognitive obstacles involved in perceiving an equation as a mathematical object on which they can perform operations.

Some of these obstacles include a limited view of the equal sign, the idea of equivalent equations, interpreting the structure of equations and constructing meaning for formal solution procedures.

Limited view of the equal sign

Consider the equation $2x + 3 = 11$. Some students see the expression on the left-side as a process and the expression on the right-side as the result (Linchevski & Herscovics, 1996). If algebraic expressions are seen as processes rather than objects, then the equality sign is interpreted as a "do something symbol" (Sfard & Linchevski, 1994). Several researchers have noted that such a limited view of the equal sign exists among some students in secondary school (Herscovics & Linchevski, 1994) and also at college level (Bell, 1995). This suggests that if learners have such a limited view of the 'equal-to sign' then they will find it difficult to work with equations of the type

$$4 \cos x + 1 = -1.$$

Didactic cut

Research on the solution of linear equations shows that while learners can solve equations of the type

$$ax + b = c \ldots (1),$$

they have difficulty solving equations in which the unknown appears on both sides of the equation. For example, equations of the type

$$ax + b = cx + d \ldots (2)$$

(Sfard & Linchevski, 1994). Herscovics and Linchevski (1994) citing Filloy and Rojano refer to this as a 'didactic cut' between arithmetic and algebra. In (1) the equality still functions as in arithmetic, operations on one side and the result on the other. In (2) the equal sign represents an equivalence relation. Learners who display a 'didactic cut' will experience difficulty when confronted with equations of the type .

Formal solving methods

Formal methods of equation solving, which require that an equation be treated as a mathematical object, include transposing and the performing of the same operation on both sides of the equation. While performing the same operation on both sides of an equation makes use of and emphasises the symmetry of an equation, this is absent in the procedure of transposing. Substitution is one of the methods used to solve equations. However, research suggests that once learners learn a formal method for solving an equation, they tend to drop the use of substitution for verifying the correctness of their solution (Kieran, 1992). Research evidence has also shown that learners who view equations as entities with symmetric balance find it easier to operate on the structure of an equation by performing the same operation on both sides (Kieran, 1992). This indicates that for formal equation-solving the following order of instruction could help learners treat equations as algebraic objects: first establish that the equality sign is a symbol that denotes the equivalence between the left and the right sides of an equation, followed by instruction on performing the same operation on both sides, and then instruction on the use of substitution for verifying. Further, learners must be equipped with strategies for solving different types of equations, for example, linear, quadratic and cubic equations. This requires that appropriate algorithms be developed to solve different types of equations. To use an appropriate algorithm a learner must first analyse the structure of the equation.

Structural features of equations

Kieran (1992) documented the results of various studies that provided evidence of the inability of students to distinguish structural features of linear equations. A description of these follow:

a. Studies in 1982 and 1984 by Kieran (1992) showed that beginning algebra pupils do not regard $x + 4 = 7$ and $x = 7 - 4$ as equivalent equations. This provides insight into why some learners experience difficulties when confronted with an equation of the type $\sin\theta + 2 = 1$.
b. In another study by Wagner and his colleagues in 1984 it was indicated that some high school students did not regard $7w + 22 = 109$ and $7n + 22 = 109$ as equivalent equations.
c. With reference to a study by Wagner and his colleagues done in 1984 Kieran (1992:403) wrote

    > The findings of this study show that most algebra students have trouble dealing with multi-term expressions as a single unit and suggest that students do not perceive the basic surface structure of, for example, say $4(2r + 1) + 7 = 35$, is the same as, say, $4x + 7 = 35$.

    This provides insight into why some learners find it difficult to proceed towards a solution of an equation of the type $2\sin^2\theta - 5\sin\theta + 2 = 0$. They are unable to detect that this is a quadratic equation in and has the same basic surface structure as the quadratic equation $2k^2 - 5k + 2 = 0$.
d. A study by Greeno in 1982 noted that beginning algebra students lack knowledge of the constraints that determine whether a transformation is

permissible. For example in an equation such as $x + 1 = 5$, students were unable to use the equivalence constraint to show that an incorrect solution, say $x = 3$, is wrong. This is a finding also observed by Kieran (1992: 403) who reported that "... competent high school solvers also lacked this knowledge".

e. Citing research by Lewis and Bernard done in 1980, Kieran (1992:401) noted

> Students have generally been found to lack the ability to generate and maintain a global overview of the features of an equation that should be attended to in deciding upon the next algebraic transformation to be carried out.

These studies indicate that learners have difficulty in (i) recognising equivalent equations, (ii) interpreting the basic surface structure of equations, (iii) dealing with multi-term equations (including ones in which the unknown occurs on both sides), and (iv) decision-making with regard to which transformations are permissible and should be made in the context of the given equation. An analysis of Examiners' Reports (e.g. House of Delegates, 1992; 1995; 1996) for the Senior Certificate Examination in mathematics, written in South Africa, suggests that the above conclusions are also true for quadratic and cubic equations. As a specific area of weakness the Examiner's Report (1992:28) stated

> Attempting to solve a quadratic equation without first writing it in standard form:
>
> e.g. $(x - 3)(2x + 1) = 4$
>
> $\therefore x - 3 = 4$ or $2x + 1 = 4$.

Here learners failed to interpret the structure of the equation and falsely generalised the zero product rule to it. Research evidence indicates that a structural conception of a given equation is "... a prerequisite for the comprehension of the strategy that must be used" (Sfard & Linchevski, 1994:211). The teaching implication here is that instruction should focus on and emphasise the structural features of mathematical objects (expressions, equations, functions) and their implications.

Solving word problems that lead to equations

A number of studies (e.g. Kintsh & Greeno, 1985; Burton, 1988; Kieran, 1992; Bell, 1995) have focused on the solving of word problems. These have mainly reported that learners encounter many difficulties when they are exposed to word problems. Such difficulties were identified to relate to one or more of:

a. Comprehending the word problem.
b. Specifying and expressing relations among variables.
c. Representing the information correctly by using a table of relations.
d. Detecting and using the correct verb (for example, *is* or *exceeds*) for the problem statement.
e. Noticing the structural similarities when problems have different cover stories.

f. Correctly translating the word problem into an equation or equations containing numbers, variables and operations.
g. Interpreting the result after solving the equation that was set up.

Solving word problems is important since these continuously expose learners to the full activity of beginning with a problem, formulating the equation, *solving* this equation and then *interpreting* the result (Bell, 1995). Here the teaching implication is that learners should at some stage be exposed to word problems. Perhaps word problems should be used to introduce linear, quadratic and possibly cubic equations. A focus on formulating the problem statement and transforming it into the relevant equation may give learners a deeper insight into the structural features of equations, and the need for transforming them to equivalent equations.

Functions and Calculus

Eisenberg (1991:140) has argued that the function concept is "... one of the most difficult concepts to master in the learning of school mathematics". A possible reason for this is that symbolic notation is usually used to represent functions. The concept of a function presented in the form of algebraic symbolism is an abstract concept. Any difficulty that a learner has with the conceptualisation of the symbolic representation or the context in which symbols are used will therefore impact on his/her understanding of the function concept. Further, movement is often involved in a function concept and this is an advanced idea, since the dependent variable relative to the independent variable could be continuously changing. For example, for a two-variable function the relevant ordered pairs when plotted on the Cartesian plane give a graphical representation of the function, by lighting up the path the curve follows. There is always a relationship between two or more variables. Therefore the variable concept; itself often requiring a complex process to understand; should be well developed before functions are introduced. In the secondary school syllabus displacement, velocity, and rates of change are typical topics where functions are applied.

Research has shown that some success can be achieved by introducing the function concept in a variety of representational contexts (Eisenberg, 1991). Examples here include using visual representations in the form of arrow diagrams, tables, input–output boxes or graphs, or by using algebraic representations in the form of ordered pairs or algebraic descriptions. Dubinsky (1991) has suggested that an important way of understanding the concept of a function is to construct a process. In the case of specific examples, say $y = x^2$, an individual may respond by *constructing* in the mind a mental process which relates to the function's process. This is an example of interiorisation (where some operation or process is performed on already familiar mathematical objects) which is a prerequisite for total understanding (Sfard, 1991). By making use of function machines and function games together with calculators and computers, Widmer and Sheffield (1994) have also shown that the learning difficulties associated with the function concept can be addressed to a large extent. It may be concluded that different representations of the concept of a function in a variety of contexts, and the processes they imply, could aid the achievement of interiorisation and so promote understanding.

Studies on the teaching of calculus (e.g. Keynes & Olson, 2000; Tall, 1996) have argued that a vicious circle could be set in motion if teaching occurs without promoting/facilitating understanding. To this effect Lithner (2000:94), citing Tall, has indicated

> If the fundamental concepts of calculus [..] prove difficult to master, one solution is to focus on the symbolic routines of differentiation [..]. The problem is that such routines are just that — routine — so that students begin to find it difficult to answer questions that are conceptually challenging. The teacher compensates by setting questions on examinations that students can answer and the vicious circle of procedural teaching and learning is set in motion.

Keynes and Olson (2000) have developed a profile to get students to learn calculus better and to develop critical thinking skills. They have suggested that tasks or learning activities for the teaching of calculus should aim to develop:

a. The ability to carefully carry out computations.
b. The ability to think geometrically and conceptually.
c. The ability to explore concepts creatively.
d. The ability to work independently and with others.
e. The ability to communicate mathematical concepts clearly.

This implies that teaching approaches should include student-centred learning, instructional teamwork, students working co-operatively in small groups and the exploration of mathematical ideas using appropriate technologies. Also this indicates that the ability to work competently with algebraic expressions, equations and inequalities, as well as having an advanced concept of a function, are pre-requisites for understanding calculus.

Different representations, both internal (those that occur in the mind) and external (those that are visible to others, e.g. a sketch), play an important part in understanding mathematical concepts. Dreyfus (1991:32) has indicated that although "… it is important to have many representations of a concept, their existence … is not sufficient to allow flexible use of the concept in problem solving". However, if the various representations are correctly linked then it becomes possible to switch from one representation to another which is more efficient to use. For example, the quadratic function which is an abstract concept can have an algebraic representation, say $g(x) = x^2 + x + 1$, or a graphical representation. Eisenberg (1991), citing research by Sneldon and his colleagues, has shown that students often approach problems analytically without utilizing the visual interpretation of the givens. Learners often ignore the power of the graphical representation when faced with questions of this type. At school level both graphical and analytical arguments are acceptable. It should be noted that although the analytical argument is deductive and logical, it "… may not be appropriate for the cognitive development of the learner" (Tall 1991:3).

Translating is a process which is closely connected to switching representations. Dreyfus (1991:33) noted that one meaning of this (translating) is "… going over from one formulation of a mathematical statement to another". Ignoring the visual formulation of aspects of mathematics (where possible) could lead to learning problems. Eisenberg (1991:152) has argued that "… the unwillingness to stress the visual aspects of mathematics in general, and of functions in particular, is a serious impediment of students' learning". The teaching implications are that instruction should focus on using different representations to help learners understand concepts. Further, learners should be exposed to activities that require them to switch representations and to focus on different formulations of mathematical statements.

Conclusions and recommendations

Many procedures and processes are involved in the understanding of secondary school mathematics. Mathematics makes use of symbolic notation, which serves a dual role as an instrument of communication and thought. This special language makes it possible to represent in coded form mathematical concepts, structures and relationships. While lecturing to first-year students at the University of KwaZulu-Natal I found that a significant number of students were unable to (1) interpret the structures of mathematical objects, and (2) solve word problems. This implied that teaching should focus on and emphasise the structural features of mathematical objects (expressions, equations and functions). In developing the symbolic notation teaching should not neglect the role of Ordinary English as the link between experiences and the emergence of the symbolic notation. Perhaps word problems should be used to introduce linear, quadratic and possibly cubic equations. A focus on formulating the problem statement and transforming it into the relevant equation may give learners a deeper insight into the structural features of equations, and the need for transforming them to equivalent equations.

Research in mathematics education has indicated the need to focus on the anticipation of learning problems and needed knowledge issues before they become impediments to learners' progress (English, 2002). To make a difference in the classrooms teachers need to be exposed to research literature relevant to the teaching and learning of mathematics. Insights from such studies indicate that learners have difficulty in (a) recognising equivalent equations, (b) interpreting the basic surface structure of equations, (c) dealing with multi-term equations (including ones in which the unknown occurs on both sides), and (d) decision-making with regard to which transformations are permissible and should be made in the context of the given equation.

An educator who functions at the structural level, and ignores the fact that concepts in mathematics are first conceived operationally, is unlikely to meaningfully develop in learners an understanding of mathematical concepts. Furthermore, the educator is unlikely to appreciate the cognitive obstacles experienced by learners with regard to the formation of concepts and the achieving of understanding. Instruction should take into account the links between arithmetic and algebra, algebra and geometry, and the teaching implications from research studies in mathematics. Learners should be encouraged to seek meaning when dealing with symbolic notation representing algebraic expressions, equations, and functions. Verbalisation, visualisation, and appropriate mathematical questions all contribute to sense-making. Although structural conceptions are difficult to achieve, proper planning and appropriate instruction (taking into account how understanding occurs) could overcome many of the problems encountered by learners. Appropriate verbalisation and visualisation by the educator at opportune times could help overcome some of these problems. These views are supported by Brodie (2003) who argued that listening and planning are two multi-dimensional practices which are integral to successfully facilitating and mediating in mathematics classrooms. However, for an educator to be able to do this he/she must have a thorough understanding of the subject matter. The importance of conceptual understanding (subject knowledge) has been highlighted by a number of educationists and theorists in South Africa (Long, 2003). In many classroom contexts this has been recognised "… by its absence rather than its presence" (Long, 2003:197). These points should be noted when authorities, e.g. education departments and universities, design and reflect on the mathematics curriculum for the training of pre-service and in-service educators.

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